STAT 512: Statistical Inference **Autumn 2022**

Quiz Session 8: Revenue Maximizing Auction

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Alternative derivation for the distribution of a order statistic: Let $X_1, ..., X_n$ be some independent and identically distributed (i.i.d.) random variables.

- If $X_1, ..., X_n$ come from a *discrete* distribution with probability mass function $\mathbb{P}(X = x) = P_x, x \in \mathcal{X}$ and $\mathcal{X}$ being countable, then the joint distribution of the ordered statistics $(Y_1, ..., Y_n) = \left(X_{(1)}, ..., X_{(n)}\right)$ is given by
$$\mathbb{P}(Y_1 = y_1, ..., Y_n = y_n) = n! \cdot \prod_{i=1}^n \mathbb{P}(X = y_i) = n! \cdot \prod_{i=1}^n P_{y_i}.$$
Further, the distribution of a single order statistic $Y_j = X_{(j)}$ becomes
$$\begin{aligned}\mathbb{P}(Y_j = y_j) &= \binom{n}{j-1} \left(\sum_{x<y_j} P_x\right)^{j-1} P_{y_j} \binom{n-j+1}{n-j} \left(\sum_{x\geq y_j} P_x\right)^{n-j} \\ &= \frac{n!}{(j-1)!(n-j)!} \left(\sum_{x\leq y_j} P_x\right)^{j-1} P_{y_j} \left(\sum_{x\geq y_j} P_x\right)^{n-j}.\end{aligned}$$

- If $X_1, ..., X_n$ come from a *continuous* distribution with probability density function $f_X(x)$ and cumulative distribution function (CDF) $F_X(x)$, then the joint CDF of the ordered statistics $(Y_1, ..., Y_n) = \left(X_{(1)}, ..., X_{(n)}\right)$ is given by
$$F_{Y_1,...,Y_n}(y_1, ..., y_n) = n! \cdot F_X(y_1) \cdots F_X(y_n)$$
so that the joint density becomes
$$f_{Y_1,...,Y_N}(y_1, ..., y_n) = \frac{\partial^n}{\partial y_1 \cdots \partial y_n} F_{Y_1,...,Y_n}(y_1, ..., y_n) = n! \cdot f_X(y_1) \cdots f_X(y_n).$$
In addition, we derive the CDF of a single order statistic $Y_j = X_{(j)}$ as:
$$\begin{aligned}F_{Y_j}(y_j) = \mathbb{P}\left(X_{(j)} \leq y_j\right) &= \mathbb{P}\left(\{\text{At least } j \text{ of } \{X_1, ..., X_n\} \text{ are less than } y_j\}\right) \\ &= \sum_{k=j}^n \binom{n}{k} \left[F_X(y_j)\right]^k \left[1 - F_X(y_j)\right]^{n-k}.\end{aligned}$$
Thus, the PDF of $Y_j = X_{(j)}$ becomes
$$\begin{aligned}f_{Y_j}(y_j) &= \frac{d}{dy_j} F_{Y_j}(y_j) \\ &= \sum_{k=j}^n \frac{n!}{(n-k)!(k-1)!} \left[F_X(y_j)\right]^{k-1} \left[1 - F_X(y_j)\right]^{n-k} f_X(y_j) \\ &\quad - \sum_{k=j}^{n-1} \frac{n!}{(n-k-1)!k!} \left[F_X(y_j)\right]^k \left[1 - F_X(y_j)\right]^{n-k-1} f_X(y_j)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=j}^{n} \frac{n!}{(n-k)!(k-1)!} \left[F_X(y_j)\right]^{k-1} \left[1 - F_X(y_j)\right]^{n-k} f_X(y_j) \\
&\quad - \sum_{k=j+1}^{n} \frac{n!}{(n-k)!(k-1)!} \left[F_X(y_j)\right]^{k-1} \left[1 - F_X(y_j)\right]^{n-k} f_X(y_j) \\
&= \frac{n!}{(n-j)!(j-1)!} \left[F_X(y_j)\right]^{j-1} \left[1 - F_X(y_j)\right]^{n-j} f_X(y_j).
\end{aligned}
$$

It gives an alternative derivation for the distribution of a order statistic $Y_j = X_{(j)}$.

Application of Order Statistics in Economics: In real-world auctions, there are generally one or more items to be sold with several bidders expressing their interests in buying the item(s). Each bidder has his/her private valuation V on the item(s) and his/her associated bidding strategy follows truthfully from the distribution of V. The bidder that places the highest bid will get the item(s). However, the actual price paid by the winning bidder could be different, depending on how the auction is designed. We will study the expected revenue of the seller under different auction scenarios in the following problem.

Problem 1 (Revenue-Maximizing Single-Item Auction)**.** *We consider the simplest but most common scenario in auction theory, where there are only a single item to be sold. In the following questions, the private valuation of each bidder is a positive random variable, and no bidder knows the bid of any other participant (*i.e.*, sealed-bid auction).*

(a) (Single-Item, One-Bidder Auction) *Suppose that the private valuation V of the bidder follows the uniform distribution on $[0,1]$ and the posted price for the item to be sold is a number $p \in (0,1)$. What is the expected revenue of the seller? (Consider also the general case where V has a support $[0,M]$ with a generic CDF F for some $M > 0$.)*

(b) (Single-Item, Two-Bidders, First-Price Auction) *Now, we consider the situation where there are two bidders bidding on a single item. The valuations V_1, V_2 of two bidders are independent and follow the uniform distribution on $[0,1]$, respectively. The bidder that places a higher bid will win the item and pay the seller as his/her bid. What is the expected revenue of the seller? (Consider also the general case where V_1, V_2 are i.i.d. with their CDFs as F supporting on $[0,M]$ for some $M > 0$.)*

(c) (Single-Item, Two-Bidders, Second-Price Auction) *Again, we consider the situation where there are two bidders bidding on a single item. The valuations V_1, V_2 of two bidders are independent and follow the uniform distribution on $[0,1]$, respectively. The bidder that places a higher bid will still win the item but he/she will pay the a price equal to the second-highest bid. What is the expected revenue of the seller? (Consider also the general case where V_1, V_2 are i.i.d. with their CDFs as F supporting on $[0,M]$ for some $M > 0$.)*

(d) (Single-Item, Two-Bidders, Second-Price Auction with a reserve price) *Again, we consider the situation where there are two bidders bidding on a single item. The valuations V_1, V_2 of two bidders are independent and follow the uniform distribution on $[0,1]$, respectively. In the second-price auction with reserve price $r \in (0,1)$, the allocation rule awards the item to the highest bidder, unless all bids are less than r, in which case no one gets the item. The corresponding payment rule charges the winner (if any) the second-highest bid or r, whichever is larger. What is the expected revenue of the seller?*

(e) (Single-Item, Multiple-Bidders Auction) *Now, we generalize (b) and (c) to the situation where there are n bidders showing their interests in buying an item. Assume that the valuations $V_1, ..., V_n$ of n bidders are i.i.d. uniform on $[0,1]$. What is the expected revenue of the seller in the first-price and second-price auction, respectively?*

(f) *Assume that the valuation distribution F for each bidder has a bounded support on $[0, M]$. Let R_a, R_b be the revenues of the seller under conditions in (a) and (b), respectively. Prove that*

$$\mathbb{E}(R_b) \geq \max_{p \in [0,M]} \mathbb{E}(R_a),$$

i.e., the expected revenue from a single-item, two-bidders auction is always higher than the expected revenue from a single-item, single-bidder auction.

(g*) *Assume that the valuation distribution F for each bidder has a bounded support on $[0, M]$ with a non-degenerate density f and the function $g(x) \equiv x[1 - F(x)]$ is concave within $[0, M]$. Let R_a, R_b, R_c, R_d be the revenues of the seller under conditions in (a), (b), (c), (d), respectively. Prove that*

$$\mathbb{E}(R_b) \geq \max_{r \in [0,M]} \mathbb{E}(R_d) \geq \mathbb{E}(R_c) \geq \max_{p \in [0,M]} \mathbb{E}(R_a),$$

where $\max_{r \in [0,M]} \mathbb{E}(R_d)$ is the maximal expected revenue under the single-item, two-bidders, second-price auction with respect to the reserve price r, and $\max_{p \in [0,M]} \mathbb{E}(R_a)$ is the maximal expected avenue under the single-item, one-bidder auction with respect to the posted price p.

Remark 1. *We emphasize that the concavity assumption on $g(x) \equiv x[1 - F(x)]$ is indeed a mild condition. This is because $g''(x) = -2f(x) - xf'(x)$ when the density f is differentiable. As long as the density f is non-decreasing or $|xf'(x)| \leq 2f(x)$ for any $x \in [0, M]$, the assumption holds.*

Solution.

(a) We consider the general case where V has a generic CDF F on $[0, \infty)$. The revenue R_a of the seller follows

$$R_a = \begin{cases} p, & \text{if } V \geq p, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, $\mathbb{E}(R_a) = \int_p^M p \, dF(v) = p\,[1 - F(p)]$. When F is Uniform$[0, 1]$, $\mathbb{E}(R_a) = p(1 - p)$, whose maximum value is $\frac{1}{4}$ at $p = \frac{1}{2}$.

(b) When both the valuation V_1, V_2 have their CDFs as F on $[0, \infty)$ and are independent, the revenue R_b of the seller is

$$R_b = \begin{cases} V_1, & \text{if } V_1 \geq V_2, \\ V_2, & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} \mathbb{E}(R_b) &= \int_{\{(v_1,v_2) \in [0,M]^2 : v_1 \geq v_2\}} v_1 \, dF(v_1) dF(v_2) + \int_{\{(v_1,v_2) \in [0,M]^2 : v_1 < v_2\}} v_2 \, dF(v_1) dF(v_2) \\ &= 2 \int_0^M x F(x) \, dF(x). \end{aligned}$$

When F is Uniform$[0, 1]$, $\mathbb{E}(R_b) = 2 \int_0^1 x^2 \, dx = \frac{2}{3}$.

(c) Under the second-price auction, the revenue R_c of the seller becomes

$$R_c = \begin{cases} V_2, & \text{if } V_1 \geq V_2, \\ V_1, & \text{if } V_1 < V_2. \end{cases}$$

Then,

$$\begin{aligned}\mathbb{E}(R_c) &= \int_{v_1 \geq v_2} v_2 \, dF(v_1)dF(v_2) + \int_{v_1 < v_2} v_1 \, dF(v_1)dF(v_2) \\ &= 2\int_0^M x\,[1 - F(x)]\, dF(x).\end{aligned}$$

When F is Uniform$[0,1]$, $\mathbb{E}(R_c) = 2\int_0^1 x(1-x)dx = \frac{1}{3}$.

(d) If the seller introduces a reserve price into the second auction, his/her revenue R_d becomes

$$R_d = \begin{cases} \max(V_2, r), & \text{if } V_1 \geq V_2 \text{ and } V_1 \geq r, \\ \max(V_1, r), & \text{if } V_1 < V_2 \text{ and } V_2 \geq r, \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned}\mathbb{E}(R_d) &= \int_r^M \int_0^{v_1} \max(v_2, r)\, dF(v_2)dF(v_1) + \int_r^M \int_0^{v_2} \max(v_1, r)\, dF(v_1)dF(v_2) \\ &= 2\int_r^M \int_0^{v_1} \max(v_2, r)\, dF(v_2)dF(v_1).\end{aligned}$$

When F is Uniform$[0,1]$ and $r \in (0,1)$,

$$\begin{aligned}\mathbb{E}(R_d) &= 2\int_r^1 \int_0^{v_1} \max(v_2, r)\, dv_2 dv_1 \\ &= 2\int_r^1 \int_r^{v_1} v_2\, dv_2 dv_1 + 2\int_r^1 \int_0^r r\, dv_2 dv_1 \\ &= \int_r^1 (v_1^2 - r^2)\, dv_1 + \int_r^1 2r^2\, dv_1 \\ &= -\frac{4r^3}{3} + r^2 + \frac{1}{3},\end{aligned}$$

which is maximized at $r = \frac{1}{2}$ with maximum value as $\max_{r\in(0,1)} \mathbb{E}(R_d) = \frac{5}{12}$; see also Figure 1.

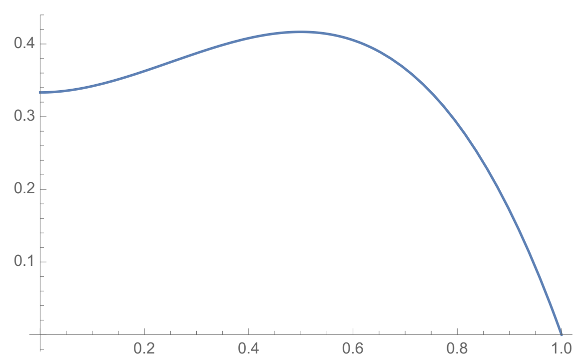

Figure 1: Plot of $-\frac{4r^3}{3} + r^2 + \frac{1}{3}$ on $r \in [0,1]$.

(e) Under the i.i.d. Uniform$[0,1]$ assumption on $V_1, ..., V_n$, we denote their order statistics by $V_{(1)} \leq V_{(2)} \leq \cdots \leq V_{(n)}$. By the notes in Lecture 9, we know that $Y_i = V_{(i)}$ has the density function as:

$$f_{Y_i}(y) = \frac{n!}{(i-1)!(n-i)!} \cdot y^{i-1}(1-y)^{n-i}, \quad 0 < y < 1,$$

which is a Beta distribution Beta$(i, n-i+1)$. The revenue R_e of the seller is $R_e = Y_{(n)}$ for the first-price auction and $R_e = Y_{(n-1)}$ for the second-price auction. Hence, based on the properties of the beta distribution,

$$\mathbb{E}(R_e) = \begin{cases} \frac{n}{n+1} & \text{for the first-price auction,} \\ \frac{n-1}{n+1} & \text{for the second-price auction.} \end{cases}$$

Indeed, we can also compute the variance or other moments of the revenue R_e, given that its distribution has an analytic solution.

(f) Based on (a) and (b), we know that

$$\begin{aligned}
\mathbb{E}(R_b) - \max_{p\in[0,M]} \mathbb{E}(R_a) &= 2\int_0^M xF(x)dF(x) - \max_{p\in[0,M]} p\left[1-F(p)\right] \\
&= \int_0^M x\, d\left[F(x)\right]^2 - p^*\left[1-F(p^*)\right] \\
&= x\left[F(x)\right]^2 \Big|_0^M - \int_0^M F(x)^2 dx - p^*\left[1-F(p^*)\right] \\
&\geq M - \int_0^M F(x)\, dx - p^*\left[1-F(p^*)\right] \qquad \text{since } 0 \leq F(x) \leq 1 \\
&= M - \int_0^{p^*} F(x)\, dx - \int_{p^*}^M F(x)\, dx - p^*[1-F(p^*)] \\
&\geq M - \int_0^{p^*} F(p^*)\, dx - \int_{p^*}^M dx - p^*[1-F(p^*)] \\
&= 0,
\end{aligned}$$

where p^* is a maximum point of $p[1-F(p)]$ over $[0,1]$ (not necessarily unique). The result follows.

(h) We first show that $\mathbb{E}(R_b) \geq \max_{r\in[0,M]} \mathbb{E}(R_d)$. Recall from (b) and (d) that

$$\begin{aligned}
\mathbb{E}(R_b) - \max_{r\in[0,M]} \mathbb{E}(R_d) &= 2\int_0^M \int_0^{v_1} v_1\, dF(v_2)dF(v_1) - 2\int_{r^*}^M \int_0^{v_1} \max(v_2, r^*)\, dF(v_2)dF(v_1) \\
&\geq 2\int_{r^*}^M \int_0^{v_1} v_1\, dF(v_2)dF(v_1) - 2\int_{r^*}^M \int_0^{v_1} \max(v_2, r^*)\, dF(v_2)dF(v_1) \\
&= 2\int_{r^*}^M \int_0^{v_1} \left[v_1 - \max(v_2, r)\right] dF(v_2)dF(v_1) \\
&\geq 0,
\end{aligned}$$

where $r^* = \arg\max_{r\in[0,M]} \mathbb{E}(R_d)$ (not necessarily unique). Hence, we obtain that $\mathbb{E}(R_b) \geq \max_{r\in[0,M]} \mathbb{E}(R_d)$.

Now, we consider proving $\max_{r\in[0,M]} \mathbb{E}(R_d) \geq \mathbb{E}(R_c)$. It suffices to show that there exists some $r \in [0, M]$ such that this inequality holds. From (c) and (d), we obtain that

$$\begin{aligned}
h(r) &\equiv \mathbb{E}(R_d) - \mathbb{E}(R_c) \\
&= 2\int_r^M \int_0^{v_1} \max(v_2, r)\, dF(v_2)dF(v_1) - 2\int_0^M \int_0^{v_1} v_2\, dF(v_2)dF(v_1) \\
&= 2\int_r^M \int_0^r r\, dF(v_2)dF(v_1) + 2\int_r^M \int_r^{v_1} v_2\, dF(v_2)dF(v_1) - 2\int_0^M \int_0^{v_1} v_2\, dF(v_2)dF(v_1)
\end{aligned}$$

$$= 2rF(r)\left[1 - F(r)\right] + 2\int_r^M g(v_1, r)\, dF(v_1) - 2\int_0^M \int_0^{v_1} v_2\, dF(v_2) dF(v_1),$$

where $g(v_1, r) = \int_r^{v_1} v_2\, dF(v_2) = \int_r^{v_1} v_2 f(v_2)\, dv_2$. Since $h(0) = 0$, we only need to show that $h(r)$ is (strictly) increasing in $[0, \delta]$ for some $\delta > 0$. Then, after optimizing $\mathbb{E}(R_d)$ with respect to $r \in [0, M]$, its value is guaranteed to be greater than $\mathbb{E}(R_c)$. As F has its density f, we differentiate $h(r)$ as

$$\begin{aligned}
h'(r) &= \left[2F(r) + 2rf(r)\right]\left[1 - F(r)\right] - 2rF(r)f(r) - 2g(r, r) \cdot f(r) + 2\int_r^M \frac{\partial}{\partial r} g(v_1, r)\, dF(v_1) \\
&= \left[2F(r) + 2rf(r)\right]\left[1 - F(r)\right] - 2rF(r)f(r) - 2\int_r^M rf(r)\, dF(v_1) \\
&= \left[2F(r) + 2rf(r)\right]\left[1 - F(r)\right] - 2rF(r)f(r) - 2rf(r)\left[1 - F(r)\right] \\
&= 2F(r)\left[1 - F(r) - rf(r)\right].
\end{aligned}$$

Notice that $F(r) = \int_0^r f(x)\, dx \in [0, 1]$ is integrable, so $\lim_{r \to 0} rf(r) = 0$ (recall the ratio test). Given that the density f is non-degenerate, we can always find some $\delta > 0$ such that $F(\delta) > 0$ and $1 - F(\delta) > \delta f(\delta)$, which in turn shows that $h'(\delta) > 0$. Hence, $h(\delta) > 0$ for some $\delta > 0$ and the result follows.

Lastly, we will prove that $\mathbb{E}(R_c) \geq \max_{p \in [0,M]} \mathbb{E}(R_a)$. Under the concavity assumption on $g(x) \equiv x[1 - F(x)]$,

$$g(\alpha p^*) \geq \alpha g(p^*) + (1 - \alpha)g(0) \quad \text{and} \quad g\left(\alpha p^* + (1 - \alpha)M\right) \geq \alpha g(p^*) + (1 - \alpha)g(M)$$

for any $\alpha \in [0, 1]$, where p^* is a maximum point of $p[1 - F(p)]$ over $[0, 1]$ (not necessarily unique). It indicates that the shape of $g(x) \equiv x[1 - F(x)]$ has its area larger than the triangle with vertices $(0, 0)$, $(p^*, g(p^*))$, and $(M, 0)$. Therefore,

$$\begin{aligned}
\mathbb{E}(R_c) - \max_{p \in [0,M]} \mathbb{E}(R_a) &= 2\int_0^M x\left[1 - F(x)\right] dF(x) - \max_{p \in [0,M]} p[1 - F(p)] \\
&\geq 2 \cdot \frac{1}{2} \cdot p^*[1 - F(p^*)] \cdot [F(M) - F(0)] - p^*[1 - F(p^*)] \\
&= 0.
\end{aligned}$$

The results follow. □

Remark 2. *The above problem reveals several interesting insights into the auction process. First, for the single-item auction, the revenue increases as more bidders enter the bidding process. In other words, the competition between bidders creates more revenue for the seller. Second, the first-price auction generally yields more revenue than the second-price auction, but it is more difficult for the seller to predict the bidding outcome and control this process [Roughgarden, 2016]. More importantly, some advanced bidders can leverage the bid shading*[1] *to lower their bids and hunt for a higher payoff after multiple rounds of the first-price bidding. It leads to a malicious trading environment for both the seller and other less experienced bidders (especially under the online advertising scenario). On the other hand, in the second-price (or Vickrey) auction, the actual payment for the winning bidders is equal to the second highest bid, which is generally unknown to all the bidders. Further, the second-price auction embraces many compelling properties (see Chapter 2 in Roughgarden 2016):*

- *Every bidder has a dominant strategy*[2]*: set the bid equal to his/her private valuation.*
- *Every truthful bidder is guaranteed nonnegative utility.*

[1] See `https://en.wikipedia.org/wiki/Bid_shading`.

[2] A dominant strategy is defined as a strategy (i.e., a bid) that is guaranteed to maximize a bidder's utility, no matter what the other bidders do.

Due to these two properties, the single-item, second-price auction is dominant-strategy incentive compatible *(DSIC). The second-price auction and its generalized version is widely used in the digital/online advertising industry.*

References

T. Roughgarden. *Twenty lectures on algorithmic game theory.* Cambridge University Press, 2016.