MATRICES WITH PRESCRIBED OFF-DIAGONAL ELEMENTS

BY

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ABSTRACT

It is shown that there exists an $n\times n$ matrix all of whose eigenvalues and off-diagonal elements are prescribed. The number of such matrices is finite.

1. Introduction

It was shown by Mirsky [2] that for any $2n-1$ complex numbers $\lambda_1,\cdots,\lambda_n$, $b_1,\cdots,b_{n-1}$ there exists an n-square matrix with eigenvalues $\lambda_1,\cdots,\lambda_n$ and $n-1$ its main diagonal entries equal to $b_1,\cdots,b_{n-1}$. Recently it was proved by D. London and H. Minc [1] that the restriction of the $n-1$ prescribed entries to the main diagonal is unnecessary. In this paper we show that all n^2-n off-diagonal elements can be prescribed, that is, there exists a matrix with these elements all of whose eigenvalues are prescribed. We also show that the number of $n\times n$ matrices with prescribed eigenvalues and all off-diagonal elements is finite. Similar results hold for other Schur matrix functions. This is done by proving preliminary theorems on solvability of certain systems of polynomials equations that may be of interest by itself.

2. Preliminary results

Denote by $K[x_1,\cdots,x_n]$ the polynomial ring of n indeterminates over an algebraically closed field K. Let

$$f_i(x_1,\cdots,x_n)=x_i^{m_i}+P_i(x_1,\cdots,x_n),\qquad i=1,\cdots,n, \tag{1}$$

where

$$\deg P_i(x_1,\cdots,x_n)<m_i,\quad i=1,\cdots,n.$$

Denote by $I[f_1,\cdots,f_n]$ the ideal generated by $f_1,\cdots,f_n$.

* Received July 26, 1971

THEOREM 1. *Let $P(x_1, \cdots, x_n)$ be a nonzero polynomial of the form*

$$P(x_1, \cdots, x_n) = \sum a_{i_1 \ldots i_n} x_1^{i_1} \cdots x_n^{i_n}, \qquad 0 \leqq i_1 < m_1, \cdots, 0 \leqq i_n < m_n \tag{2}$$

then $P(x_1, \cdots, x_n)$ is not in $I[f_1, \cdots, f_n]$.

PROOF. First we notice that every polynomial $Q(x_1, \cdots, x_n)$ of n indeterminates $x_1, \cdots, x_n$ can be written as a polynomial expression $Q^*(x_1, \cdots, x_n, f_1, \cdots, f_n)$ of $x_1, \cdots, x_n, f_1, \cdots, f_n$, that is a sum of monomials of the form

$$A x_1^{j_1} \cdots x_n^{j_n} f_1^{s_1} \cdots f_n^{s_n}, \qquad 0 \leqq j_1 < m_1, \cdots, 0 \leqq j_n < m_n. \tag{3}$$

This can be done in a finite number of steps: for each $x_i^{m_i t_i + q_i} (0 \leqq t_i, 0 \leqq q_i < m_i)$ we substitute the expression $(f_i - P_i)^{t_i} x_i^{q_i}$ which is of a lower degree in the x_i. The polynomial $Q^*(x_1, \cdots, x_n, f_1, \cdots, f_n)$ is uniquely determined. To prove this it is enough to show that every nonzero polynomial $Q^*(x_1, \cdots, x_n, f_1, \cdots, f_n)$, i.e., a sum of monomials of the form (3), is transformed by substitutions (1) to a nonzero polynomial $\tilde{Q}(x_1, \cdots, x_n)$. From all terms of $Q^*(x_1, \cdots, x_n, f_1, \cdots, f_n)$ pick a term of the form (3) for which the sum

$$(s_1 m_1 + j_1) + (s_2 m_2 + j_2) + \cdots + (s_n m_n + j_n)$$

is maximal. Denote this maximal sum by m. Now, using Eqs. (1) we find that the degree of $\tilde{Q}(x_1, \cdots, x_n)$ is not greater than m. Let us compute the coefficient of a monomial

$$x_1^{s_1 m_1 + j_1} \cdots x_n^{s_n m_n + j_n}$$

of degree m appearing in $\tilde{Q}(x_1, \cdots, x_n)$. Since a sum $(s_1 m_1 + j_1) + \cdots + (s_n m_n + j_n)$ is maximal this term came only from an appropriate term

$$x_1^{j_1} \cdots x_n^{j_n} f_1^{s_1} \cdots f_n^{s_n}; \qquad 0 \leqq j_1 < m_1, \cdots, 0 \leqq j_n < m_n,$$

in $Q^*(x_1, \cdots, x_n, f_1, \cdots, f_n)$. Thus the coefficients of the two monomials are equal and the degree of $\tilde{Q}(x_1, \cdots, x_n)$ is exactly m.

Now it is clear that $Q(x_1, \cdots, x_n)$ belongs to $I[f_1, \cdots, f_n]$ if and only if $Q^*(x_1, \cdots, x_n, f_1, \cdots, f_n)$ is a sum of monomials of the form (3) satisfying the inequality:

$$1 \leqq s_1 + \cdots + s_n.$$

We observe that for a polynomial $P(x_1, \cdots, x_n)$

$$P(x_1, \cdots, x_n) = \sum a_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}; \qquad 0 \leqq i_1 < m_1, \cdots, 0 \leqq i_n < m_n$$

we have the identity

$$P(x_1, \cdots, x_n) = P^*(x_1, \cdots, x_n, f_1, \cdots, f_n)$$

Since P^* is unique we can conclude that $P(x_1, \cdots, x_n)$ cannot belong to $I[f_1, \cdots, f_n]$.

THEOREM 2. *The n polynomial equations*

$$x_i^{m_i} + P_i(x_1, \cdots, x_n) = 0, \qquad i = 1, \cdots, n, \tag{4}$$

where the degree of each $P_i(x_1, \cdots, x_n)$ is smaller than m_i, are solvable over an algebraically closed field K. The number of solutions of the system (4) is finite.

PROOF. The solvability of system (4) follows immediately from Hilbert's Nullstellensatz [5] and from Theorem 1. Suppose that $\xi = (\xi_1, \cdots, \xi_n)$ is a solution of system (4). Then each ξ_i satisfies

$$\xi_i^{m_i} = -P_i(\xi_1, \cdots, \xi_n), \qquad i = 1, \cdots, n.$$

Since the degree of $P_i(\xi_1, \cdots, \xi_n)$ is less than m_i every polynomial $Q(\xi_1, \cdots, \xi_n)$ can be expressed as a polynomial of the form

$$Q(\xi_1, \cdots, \xi_n) = \sum a_{i_1 \cdots i_n} \xi_1^{i_1} \cdots \xi_n^{i_n}, \qquad 0 \leqq i_1 < m_1, \cdots, 0 \leqq i_n < m_n$$

Thus the polynomial ring $K[\xi_1, \cdots, \xi_n]$ is at most of dimension $m = m_1 m_2 \cdots m_n$. Therefore the $m + 1$ elements

$$1, \xi_i, \xi_i^2, \cdots, \xi_i^m$$

are linearly dependent. That means that each ξ_i satisfies a nontrivial polynomial equation:

$$a_0^{(i)} \xi_i^m + a_1^{(i)} \xi_i^{m-1} + \cdots + a_m^{(i)} = 0, \qquad i = 1, \cdots, n.$$

This concludes the proof of Theorem 2.

3. Main result

Let $A = (a_{ij})_1^n$ be an n-square matrix over an algebraically closed field K. Suppose that the $n^2 - n$ off-diagonal elements are prescribed. Let $x_i = a_{ii}$, $i = 1, \cdots, n$ and denote the n elementary symmetric polynomials of $x_1, \cdots, x_n$ by $\sigma_1, \cdots, \sigma_n$.

Let $\lambda_1, \cdots, \lambda_n$ be prescribed eigenvalues of A. The existence of such a matrix A is equivalent to the solvability of n polynomial equations:

$$\sigma_i(x_1, \cdots, x_n) + R_i(x_1, \cdots, x_n) = \sigma_i(\lambda_1, \cdots, \lambda_n), \qquad i = 1, \cdots, n,$$

where the degree of each $R_i(x_1, \cdots, x_n)$ is smaller than i (in fact for $i \geqq 2$ the degree of each R_i is not greater than $i - 2$).

We prove the following result:

THEOREM 3. *The system of n polynomial equations of the form*

$$\sigma_i(x_1, \cdots, x_n) + Q_i(x_1, \cdots, x_n) = 0, \qquad i = 1, \cdots, n, \tag{5}$$

where the degree of each Q_i is less than i, is solvable over an algebraically closed field K. The number of solutions of the system (5) *is finite.*

PROOF. We notice first that each x_i satisfies

$$x_i^n - \sigma_1 x_i^{n-1} + \sigma_2 x_i^{n-2} + \cdots + (-1)^n \sigma_n = 0 \quad , \quad i = 1, \cdots, n. \tag{6}$$

Define

$$g_i = (-1)^{i-1}(\sigma_i + Q_i) \; , \qquad i = 1, \cdots, n$$

and

$$f_i = x_i^n + Q_1 x_i^{n-1} - Q_2 x_i^{n-2} + \cdots + (-1)^{n-1} Q_n, \qquad i = 1, \cdots, n.$$

Clearly each f_i is a polynomial of the form

$$f_i(x_1, \cdots, x_n) = x_i^n + P_i(x_1, \cdots, x_n)$$

such that the degree of P_i is less than n. Using the definitions of $g_1, \cdots, g_n, f_1, \cdots, f_n$ and the equations (6) we have the equalities

$$f_i = x_i^{n-1} g_1 + x_i^{n-2} g_2 + \cdots + g_n, \qquad i = 1, \cdots, n. \tag{7}$$

In order to express $g_1, \cdots, g_n$ as functions of $f_1, \cdots, f_n$ we compute the inverse of the matrix $(x_i^{n-j})_{i,j=1}^n$ whose determinant is equal to the Vandermonde determinant

$$W = \prod_{1 \leqq i < j \leqq n} (x_i - x_j).$$

Then the polynomial Wg_k:

$$Wg_k = \sum_{r=1}^{n} u_{kr} f_r,$$

where each u_{kr} is a polynomial in $x_1, \cdots, x_n$, belongs to $I[f_1, \cdots, f_n]$ for $k = 1, \cdots, n$. Assume, to the contrary, that the n polynomials $g_1, \cdots, g_n$ have no root in common.

From Hilbert's Nullstellensatz we deduce that

$$1 = \sum_{i=1}^{n} v_i g_i,$$

where $v_1, \cdots, v_n$ are polynomials over the field K. If we multiply both sides of this equality by W we see that the polynomial W belongs to the ideal $I[f_1, \cdots, f_n]$.

This contradicts Theorem 1. From Theorem 2 we can also conclude that the number of different solutions of the systems (5) is finite, since from (7) each f_i belongs to $I[g_1 \cdots g_n]$.

The following result follows immediately.

THEOREM 4. *Let a_{ij} $i \neq j$, $i,j = 1,\cdots,n$, and $\lambda_1,\cdots,\lambda_n$ be prescribed elements from an algebraically closed field K. Then there exists an n-square matrix over this field with eigenvalues $\lambda_1,\cdots,\lambda_n$ and whose (i,j) entry is equal to a_{ij}, $i \neq j$, $i,j = 1,\cdots,n$. The number of such matrices is finite.*

4. Remarks

In [3] De Oliveira extended Mirsky's result to permanents. He showed that there exists an $n \times n$ matrix A with prescribed $n-1$ diagonal elements and prescribed polynomial per $(\lambda I - A)$. We generalize Theorem 4 in a similar manner to a result involving Schur matrix functions.

Let G be a subgroup of S_n, the symmetric group of degree n, and let χ be a character of G over an algebraically closed field. K, i.e., a notrivial multiplicative homomorphism of G into K. The generalized matrix function d_χ^G, introduced by I. Schur [4], associated with G and χ is a function on n-square matrices over K whose value for the matrix $X = (x_{ij})_1^n$ is defined by the formula

$$d_\chi^G(X) = \sum_{\sigma \in G} \chi(\sigma) \sum_{i=1}^{n} x_{i\sigma(i)}.$$

Define the generalized characteristic roots $\lambda_1,\cdots,\lambda_n$ of matrix A associated with tbe function d_χ^G as the roots of the polynomial

$$d_\chi^G(\lambda I - A) = 0$$

where I is the unit matrix of order n.

Suppose that all generalized characteristic roots and $n^2 - n$ off-diagonal elements of an n-square matrix are given. The product

$$(\lambda - a_{11})(\lambda - a_{22})\cdots(\lambda - a_{nn})$$

appears in the equation $d_\chi^G(\lambda I - A)$ for any group G and any character χ. The existence of a matrix with prescribed $n^2 - n$ off-diagonal elements and prescribed generalized roots is equivalent to the solvability of n polynomial equations of the form (5). According to Theorem 3 we have

THEOREM 5. *Let a_{ij}, $i \neq j$, $i,j = 1,\cdots,n$ and $\lambda_1,\cdots,\lambda_n$ be prescribed elements* *from an algebraically closed field K. Then there exists an n-square matrix over this field with generalized eigenvalues* $\lambda_1, \cdots, \lambda_n$ *and whose* (i,j) *entry is equal to* a_{ij}, $i \neq j$, $i,j = 1, \cdots, n$. *The number of such matrices is finite**.

References

1. D. London and H. Minc, *Eigenvalues of matrices with prescribed entries*, Technion Preprint, MT–80, May 1971.

2. L. Mirsky, *Matrices with prescribed characteristic roots and diagonal elements*, J. London Math. Soc. **33** (1958), 14–21.

3. G. N. de Oliveira, *A conjecture and some problems on permanents*, Pacific J. Math. **32** (1970), 495–499.

4. I. Schur, *Über endliche gruppen und hermitesche formen*, Math. Z. **1** (1918), 184–201.

5. Van der Waerden, *Modern Algebra*, Vol. II, Fredrick Ungar Publishing Co., 1950.

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* The author wishes to thank Professors D. London and H. Minc for their helpful remarks.