



Poincaré series of some pure and mixed trace algebras of two generic matrices

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Abstract

We work over a field K of characteristic zero. The Poincaré series for the algebra $C_{n,2}$ of GL_n -invariants and the algebra $T_{n,2}$ of GL_n -concomitants of two generic $n \times n$ matrices x and y are computed for $n \leq 6$. Both simply graded and bigraded cases are included. The cases $n \leq 4$ were known previously. For $C_{4,2}$ and $C_{5,2}$ we construct a minimal set of generators, and give an application to Specht's theorem on unitary similarity of matrices.

By identifying the space M_n^2 of pairs of $n \times n$ matrices with $M_n \otimes K^2$, we extend the action of GL_n to $\mathrm{GL}_n \times \mathrm{GL}_2$. For $n \leq 5$, we compute the Poincaré series for the polynomial invariants of this action when restricted to the subgroups $\mathrm{GL}_n \times \mathrm{SL}_2$ and $\mathrm{GL}_n \times \Delta_1$, where Δ_1 is the maximal torus of SL_2 consisting of diagonal matrices.

Five conjectures are proposed concerning the numerators and denominators of various Poincaré series mentioned above. Some heuristic formulas and open problems are stated.

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1. Introduction

Let K be a field of characteristic 0, $M_n = M_n(K)$ the K -algebra of $n \times n$ matrices over K , and d a positive integer. The general linear group $\mathrm{GL}_n = \mathrm{GL}_n(K)$ acts on the direct product M_n^d

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of d copies of M_n by simultaneous conjugation

$$a \cdot (x_1, \dots, x_d) = (ax_1a^{-1}, \dots, ax_da^{-1}).$$

This gives rise to an action of GL_n on the algebra $K[M_n^d]$ of polynomial functions on M_n^d . We shall view the entries of x_k , $k = 1, \dots, d$, as linear functions on M_n^d .

It is a well-known fact, due to Procesi [13] and Razmyslov [14], that the algebra $C_{n,d} = K[M_n^d]^{\mathrm{GL}_n}$ of GL_n -invariants in $K[M_n^d]$ is generated by all traces

$$\mathrm{tr}(z_1 z_2 \cdots z_k), \quad z_1, z_2, \dots, z_k \in \{x_1, \dots, x_d\}, \quad k \geq 1.$$

This assertion remains valid if one imposes the restriction $k \leq n^2$. For these facts and many other known properties of $C_{n,d}$ we refer the reader to one of Refs. [4,6,9] and the papers quoted there.

Similarly, GL_n acts on the (noncommutative) algebra of polynomial maps $M_n^d \rightarrow M_n$. Its subalgebra consisting of GL_n -equivariant maps will be denoted by $T_{n,d}$. The algebra $C_{n,d}$ respectively $T_{n,d}$ is known as the *pure* respectively *mixed trace algebra* of d generic $n \times n$ matrices.

By assigning the degree $(1, 0, \dots, 0)$ to the entries of the matrix x_1 , the degree $(0, 1, 0, \dots, 0)$ to the entries of x_2 , etc., one obtains a \mathbf{Z}^d -gradation of the algebras $C_{n,d}$ and $T_{n,d}$. The total degree provides these algebras with the ordinary \mathbf{Z} -gradation. In this paper we are mostly interested in the case $d = 2$. Until further notice, we assume that this is the case and we set $x = x_1$ and $y = x_2$. The simply graded and bigraded Poincaré series for both $C_{n,2}$ and $T_{n,2}$ have been explicitly computed for $n \leq 4$ (see [1,17–19]).

In Section 2 respectively 3 we give the bigraded respectively simply graded Poincaré series of $C_{n,2}$ and $T_{n,2}$ for $n = 5, 6$. For the bigraded case see Theorem 2.2 and Tables 1 and 2. For the simply graded case see Tables 3 and 4 (and Appendices B and C, respectively). We also show (see Proposition 2.1) that the \mathbf{Z}^2 -graded algebra $C_{n,2}$ has no bigraded system of parameters if $n = 5, 6$. The numerators of the bigraded Poincaré series of $C_{6,2}$ and $T_{6,2}$ have 1169 and 854 terms, respectively. For that reason we do not list them in the paper. They will be posted elsewhere.

In Section 4 we construct a minimal set of generators (MSG) of $C_{4,2}$ (see Theorem 4.2) which consists of 32 elements of the form $\mathrm{tr}(w(x, y))$, where $w(x, y)$ is a word in the matrices x and y . This problem has been already solved by Drensky and Sadikova [5], but their generators are not of this simple form. However, they claim that their choice is better suited for finding a presentation of $C_{4,2}$, which is apparently still an open problem. We give an application to the problem of unitary similarity of two complex 4×4 matrices.

In Section 5 we construct an MSG, \mathcal{P} , for the algebra $C_{5,2}$. It consists of 173 bihomogeneous polynomials.

In Section 6 we identify the space M_n^d with the tensor product $M_n \otimes K^d$ and extend the action of GL_n to $\mathrm{GL}_n \times \mathrm{GL}_d$ by letting GL_d act on K^d by multiplication. We denote by $C_{n,d}^\#$ the subalgebra of $K[M_n^d]$ consisting of $\mathrm{GL}_n \times \mathrm{SL}_d$ -invariant functions. In Table 5 we record the Poincaré series of $C_{n,2}^\#$, $n \leq 5$.

In Section 7 we restrict the action of $\mathrm{GL}_n \times \mathrm{SL}_d$ to $\mathrm{GL}_n \times \Delta_{d-1}$, where Δ_{d-1} is the maximal torus of SL_d consisting of diagonal matrices. We denote by $C_{n,d}^\bullet$ the subalgebra of $K[M_n^d]$ consisting of $\mathrm{GL}_n \times \Delta_{d-1}$ -invariant functions. In Table 6 we record the Poincaré series for the algebras $C_{n,2}^\bullet$, $n \leq 5$.

In Section 8 we propose four conjectures concerning the numerators and denominators of the Poincaré series of $C_{n,2}$ and $T_{n,2}$, one more conjecture about the series for $C_{n,2}^\#$ and $C_{n,2}^\bullet$, and state an open problem.

Appendix A gives some details concerning the verification that our expression for $P(C_{5,2}; s, t)$ agrees with Formanek’s expansion in terms of Schur functions (see [1]). Appendix B contains the table of some low degree coefficients of the Taylor expansions of $P(C_{n,2}; t)$ for $n \leq 12$. We also make a couple of interesting observations, and indulge in some speculative thinking. Appendix C treats in the same way the Taylor expansions of $P(T_{n,2}; t)$ for $n \leq 6$. We warn the reader that several assertions and formulae that appear in the last two appendices are of hypothetical character and are included there only as a suggestion deserving further consideration and study.

2. Bigraded Poincaré series

Let $P(C_{n,2}; s, t)$ respectively $P(T_{n,2}; s, t)$ denote the bigraded Poincaré series of $C_{n,2}$ respectively $T_{n,2}$. These series are given by symmetric rational functions in the two variables s and t . We can write them in lowest terms as

$$P(C_{n,2}; s, t) = \frac{N(C_{n,2}; s, t)}{D(C_{n,2}; s, t)}, \quad P(T_{n,2}; s, t) = \frac{N(T_{n,2}; s, t)}{D(T_{n,2}; s, t)}, \tag{2.1}$$

where we normalize N and D by demanding that they both have constant term 1.

These Poincaré series have been computed by Teranishi [17–19] for $n \leq 4$. The formulae for the case $n = 4$ were computed independently by Berele and Stembridge [1]. They corrected several misprints in Teranishi’s formulas in [18].

We have computed explicit formulae for the numerators and denominators for all $n \leq 6$. For $n \leq 5$, the numerators $N(C_{n,2}; s, t)$ respectively $N(T_{n,2}; s, t)$ are given in Table 1 respectively 2. For $n = 6$, the numerators are huge and are posted on the arXiv. These numerators are symmetric polynomials in s and t , and they satisfy the functional equation

$$(st)^d N(C_{n,2}; s^{-1}, t^{-1}) = N(C_{n,2}; s, t),$$

where d is the degree of $N(C_{n,2}; s, t)$ as a polynomial in s . For that reason there is no need to write all the terms of $N(C_{n,2}; s, t)$. This remark also applies to $N(T_{n,2}; s, t)$.

Proposition 2.1. *$C_{5,2}$ and $C_{6,2}$ have no bigraded system of parameters.*

Proof. Assume that $C_{5,2}$ has a bigraded system of parameters, say $\{P_1, \dots, P_{26}\}$. Since $C_{5,2}$ is a Cohen–Macaulay algebra, it is a free graded module over the polynomial algebra $K[P_1, \dots, P_{26}]$. Consequently, this module has a free bigraded basis, say $\{Q_1 = 1, Q_2, \dots, Q_m\}$. It follows that

$$P(C_{5,2}; s, t) = \frac{\sum_{j=1}^m s^{e'_j} t^{e''_j}}{\prod_{i=1}^{26} (1 - s^{d'_i} t^{d''_i})} = \frac{N(C_{5,2}; s, t)}{D(C_{5,2}; s, t)},$$

Table 1
Numerators $N(C_{n,2}; s, t)$

$$N(C_{1,2}; s, t) = N(C_{2,2}; s, t) = 1, \quad N(C_{3,2}; s, t) = 1 - st + s^2t^2,$$

$$N(C_{4,2}; s, t) = (1 - st + s^2t^2)(1 - st^2 - s^2t + s^2t^3 + s^3t^2 + s^2t^4 + 2s^3t^3 + s^4t^2 + s^3t^4 + s^4t^3 - s^4t^5 - s^5t^4 + s^6t^6),$$

$$N(C_{5,2}; s, t) = 1 - st - 2st^2 - 2s^2t - st^3 - s^3t + 3s^2t^3 + 3s^3t^2 + 4s^2t^4 + 7s^3t^3 + 4s^4t^2 + 3s^2t^5 + 4s^3t^4 + 4s^4t^3 + 3s^5t^2 + s^2t^6 - 2s^3t^5 - 5s^4t^4 - 2s^5t^3 + s^6t^2 - 5s^3t^6 - 13s^4t^5 - 13s^5t^4 - 5s^6t^3 - 4s^3t^7 - 10s^4t^6 - 11s^5t^5 - 10s^6t^4 - 4s^7t^3 - 2s^3t^8 - 2s^4t^7 + 3s^5t^6 + 3s^6t^5 - 2s^7t^4 - 2s^8t^3 + 5s^4t^8 + 18s^5t^7 + 27s^6t^6 + 18s^7t^5 + 5s^8t^4 + 4s^4t^9 + 18s^5t^8 + 29s^6t^7 + 29s^7t^6 + 18s^8t^5 + 4s^9t^4 + 2s^4t^{10} + 8s^5t^9 + 12s^6t^8 + 11s^7t^7 + 12s^8t^6 + 8s^9t^5 + 2s^{10}t^4 - s^5t^{10} - 11s^6t^9 - 24s^7t^8 - 24s^8t^7 - 11s^9t^6 - s^{10}t^5 - 3s^5t^{11} - 17s^6t^{10} - 40s^7t^9 - 47s^8t^8 - 40s^9t^7 - 17s^{10}t^6 - 3s^{11}t^5 - s^5t^{12} - 10s^6t^{11} - 24s^7t^{10} - 33s^8t^9 - 33s^9t^8 - 24s^{10}t^7 - 10s^{11}t^6 - s^{12}t^5 - s^6t^{12} - s^7t^{11} + 9s^8t^{10} + 13s^9t^9 + 9s^{10}t^8 - s^{11}t^7 - s^{12}t^6 + s^6t^{13} + 10s^7t^{12} + 34s^8t^{11} + 54s^9t^{10} + 54s^{10}t^9 + 34s^{11}t^8 + 10s^{12}t^7 + s^{13}t^6 + 6s^7t^{13} + 27s^8t^{12} + 49s^9t^{11} + 62s^{10}t^{10} + 49s^{11}t^9 + 27s^{12}t^8 + 6s^{13}t^7 + 5s^8t^{13} + 10s^9t^{12} + 13s^{10}t^{11} + 13s^{11}t^{10} + 10s^{12}t^9 + 5s^{13}t^8 - 4s^8t^{14} - 19s^9t^{13} - 37s^{10}t^{12} - 45s^{11}t^{11} - 37s^{12}t^{10} - 19s^{13}t^9 - 4s^{14}t^8 - 2s^8t^{15} - 20s^9t^{14} - 48s^{10}t^{13} - 72s^{11}t^{12} - 72s^{12}t^{11} - 48s^{13}t^{10} - 20s^{14}t^9 - 2s^{15}t^8 - \dots - s^{22}t^{22} + s^{23}t^{23}.$$

Table 2
Numerators $N(T_{n,2}; s, t)$

$$N(T_{1,2}; s, t) = N(T_{2,2}; s, t) = N(T_{3,2}; s, t) = 1,$$

$$N(T_{4,2}; s, t) = 1 + s^2t^2 + s^2t^3 + s^3t^2 + s^3t^3 + s^5t^5,$$

$$N(T_{5,2}; s, t) = 1 - st^2 - s^2t + s^2t^2 + 2s^2t^3 + 2s^3t^2 + 2s^2t^4 + 5s^3t^3 + 2s^4t^2 + s^2t^5 + 2s^3t^4 + 2s^4t^3 + s^5t^2 - s^3t^6 - 3s^4t^5 - 3s^5t^4 - s^6t^3 - s^3t^7 - s^4t^6 + s^5t^5 - s^6t^4 - s^7t^3 + 4s^5t^6 + 4s^6t^5 + s^4t^8 + 6s^5t^7 + 12s^6t^6 + 6s^7t^5 + s^8t^4 + 3s^5t^8 + 6s^6t^7 + 6s^7t^6 + 3s^8t^5 + s^6t^8 + s^7t^7 + s^8t^6 - s^5t^{10} - 5s^6t^9 - 9s^7t^8 - 9s^8t^7 - 5s^9t^6 - s^{10}t^5 - 4s^6t^{10} - 9s^7t^9 - 7s^8t^8 - 9s^9t^7 - 4s^{10}t^6 - s^6t^{11} - 5s^7t^{10} - 5s^8t^9 - 5s^9t^8 - 5s^{10}t^7 - s^{11}t^6 + 3s^8t^{10} + 6s^9t^9 + 3s^{10}t^8 + 4s^8t^{11} + 6s^9t^{10} + 6s^{10}t^9 + 4s^{11}t^8 + \dots - s^{17}t^{18} - s^{18}t^{17} + s^{19}t^{19}.$$

where (d'_i, d''_i) respectively (e'_j, e''_j) is the bidegree of P_i resp Q_j . As $N(C_{5,2}; s, t)$ and $D(C_{5,2}; s, t)$ are relatively prime, we deduce that

$$\sum_{j=1}^m s^{e'_j} t^{e''_j} = N(C_{5,2}; s, t)R(s, t),$$

where $R(s, t)$ is a polynomial in s and t . By setting $s = t = 1$ in this identity, we obtain the contradiction $m = 0$ since $N(C_{5,2}; 1, 1) = 0$.

The proof for the algebra $C_{6,2}$ is similar. \square

The denominators $D(C_{n,2}; s, t)$ and $D(T_{n,2}; s, t)$ are closely related to the product

$$\Pi_n(s, t) = \prod_{i=1}^n (1 - s^i)(1 - t^i) \prod_{j=1}^{i-1} (1 - s^{i-j}t^j)^{\min(i, n+1-i)}. \tag{2.2}$$

Theorem 2.2. *For $n \leq 5$, $N(C_{n,2}; s, t)$ and $N(T_{n,2}; s, t)$ are given by Tables 1 and 2, respectively. For $n = 6$ they are posted on the arXiv [3]. For $n \leq 6$, $D(C_{n,2}; s, t)$ and $D(T_{n,2}; s, t)$ are given by*

$$\begin{aligned} D(C_{n,2}; s, t) &= \Pi_n(s, t), \quad n \leq 5; \\ D(C_{6,2}; s, t) &= (1 - st)\Pi_6(s, t); \\ D(T_{n,2}; s, t) &= (1 + s + \dots + s^{n-1})^{-1}(1 + t + \dots + t^{n-1})^{-1}D(C_{n,2}; s, t). \end{aligned}$$

Proof. By using the well-known Molien–Weyl formula (see [2]), we have:

$$\begin{aligned} P(C_{n,2}; s, t) &= \frac{1}{(1 - s)^n(1 - t)^n} \cdot \frac{1}{(2\pi i)^{n-1}} \\ &\cdot \int_{|x_1|=1} \dots \int_{|x_{n-1}|=1} \prod_{1 \leq k \leq r \leq n-1} \frac{1 - x_k x_{k+1} \dots x_r}{\varphi_{k,r}} \frac{dx_{n-1}}{x_{n-1}} \dots \frac{dx_1}{x_1}, \end{aligned}$$

where

$$\begin{aligned} \varphi_{k,r} &= (1 - s x_k x_{k+1} \dots x_r)(1 - t x_k x_{k+1} \dots x_r) \cdot \\ &\times (1 - s(x_k x_{k+1} \dots x_r)^{-1})(1 - t(x_k x_{k+1} \dots x_r)^{-1}), \end{aligned}$$

the integration is performed over the unit circles (in the counterclockwise direction), and the variables s and t have small moduli.

A similar formula is valid for $P(T_{n,2}; s, t)$. One has just to multiply the above integrand by the function

$$n + \sum_{r=1}^{n-1} \sum_{k=1}^r \left(x_k x_{k+1} \dots x_r + \frac{1}{x_k x_{k+1} \dots x_r} \right).$$

For $n = 5$ and $n = 6$ we have computed the two types of integrals by using MAPLE [12]. Each of the cases $n = 6$ required about two weeks of computing time on a machine running R10000 CPU at 250 MHz with 8 GB of RAM. \square

We have verified independently the low degree (≤ 25) coefficients in the Taylor expansions of $P(C_{5,2}; s, t)$ and $P(C_{6,2}; s, t)$ by using a formula due to Formanek (see Appendix A).

3. Simply graded Poincaré series

Let $P(C_{n,2}; t)$ denote the simply graded Poincaré series of $C_{n,2}$ and $P(T_{n,2}; t)$ the one for $T_{n,2}$. As $P(C_{n,2}; t) = P(C_{n,2}; t, t)$ and $P(T_{n,2}; t) = P(T_{n,2}; t, t)$, by setting $s = t$ in the integral formula for $P(C_{n,2}; s, t)$ respectively $P(T_{n,2}; s, t)$ one obtains a valid formula for $P(C_{n,2}; t)$ respectively $P(T_{n,2}; t)$. We can write these rational functions in lowest terms as

$$P(C_{n,2}; t) = \frac{N(C_{n,2}; t)}{D(C_{n,2}; t)}, \quad P(T_{n,2}; t) = \frac{N(T_{n,2}; t)}{D(T_{n,2}; t)}. \tag{3.1}$$

$N(C_{n,2}; t)$ and $N(C_{n,2}; t, t)$ may differ because the numerator $N(C_{n,2}; t, t)$ and the denominator $D(C_{n,2}; t, t)$ may have a common factor. Similarly, $N(T_{n,2}; t, t)$ and $D(T_{n,2}; t, t)$ may have a common factor. In Tables 3 and 4 we list the numerators and denominators for $P(C_{n,2}; t)$ and $P(T_{n,2}; t)$ for $n \leq 6$. For their Taylor coefficients see Appendices B and C, respectively.

4. Generators of the algebra $C_{4,2}$

Teranishi [17] has constructed an HSOP for the pure trace algebra $C_{4,2}$:

Theorem 4.1. *Let x and y be generic 4×4 matrices. Then the traces of the 17 matrices (which we arrange according to their degrees)*

$$\begin{aligned} &x, y; \quad x^2, xy, y^2; \quad x^3, x^2y, xy^2, y^3; \\ &x^4, x^3y, x^2y^2, xy^3, y^4, xyxy; \quad (x^2y)^2, (y^2x)^2 \end{aligned}$$

form an HSOP of the algebra $C_{4,2}$.

Table 3
Numerators $N(C_{n,2}; t)$ and denominators $D(C_{n,2}; t)$

$N(C_{1,2}; t) = N(C_{2,2}; t) = 1,$	$D(C_{1,2}; t) = (1 - t)^2,$
$D(C_{2,2}; t) = (1 - t)^2(1 - t^2)^3,$	$N(C_{3,2}; t) = 1 - t^2 + t^4,$
$D(C_{3,2}; t) = (1 - t)^2(1 - t^2)^4(1 - t^3)^4,$	
$N(C_{4,2}; t) = (1 - t^2 + t^4)(1 - t - t^3 + t^4 + 2t^5 + t^6 - t^7 - t^9 + t^{10}),$	
$D(C_{4,2}; t) = (1 - t)^3(1 - t^2)^4(1 - t^3)^5(1 - t^4)^5,$	
$N(C_{5,2}; t) = 1 + 2t - 6t^3 - 9t^4 + 2t^5 + 25t^6 + 38t^7 + 17t^8 - 34t^9 - 68t^{10}$	
$\quad - 34t^{11} + 73t^{12} + 176t^{13} + 171t^{14} + 34t^{15} - 127t^{16} - 156t^{17}$	
$\quad - 2t^{18} + 218t^{19} + 322t^{20} + 218t^{21} - \dots + 2t^{39} + t^{40},$	
$D(C_{5,2}; t) = (1 - t^2)^6(1 - t^3)^8(1 - t^4)^6(1 - t^5)^6,$	
$N(C_{6,2}; t) = 1 - 3t + 3t^2 - 3t^3 + 3t^4 + 4t^5 - 2t^6 - 8t^8 - 8t^9 + 11t^{10} + t^{11}$	
$\quad + 56t^{12} - 24t^{13} + 48t^{14} - 69t^{15} - 9t^{16} + 2t^{17} + 78t^{18} + 118t^{19}$	
$\quad + 223t^{20} + 23t^{21} + 158t^{22} - 182t^{23} + 221t^{24} - 42t^{25} + 600t^{26}$	
$\quad + 365t^{27} + 633t^{28} + 324t^{29} + 303t^{30} - 31t^{31} + 484t^{32} + 178t^{33}$	
$\quad + 1055t^{34} + 518t^{35} + 1055t^{36} + \dots - 3t^{69} + t^{70},$	
$D(C_{6,2}; t) = (1 - t)^5(1 - t^2)^3(1 - t^3)^6(1 - t^4)^9(1 - t^5)^7(1 - t^6)^7.$	

Table 4

Numerators $N(T_{n,2}; t)$ and denominators $D(T_{n,2}; t)$

$$\begin{aligned}
 N(T_{1,2}; t) &= N(T_{2,2}; t) = N(T_{3,2}; t) = 1, & D(T_{1,2}; t) &= (1-t)^2, \\
 D(T_{2,2}; t) &= (1-t)^4(1-t^2), & D(T_{3,2}; t) &= (1-t)^4(1-t^2)^4(1-t^3)^2, \\
 N(T_{4,2}; t) &= 1-t+t^3+t^5-t^7+t^8, \\
 D(T_{4,2}; t) &= (1-t)^5(1-t^2)^4(1-t^3)^5(1-t^4)^3, \\
 N(T_{5,2}; t) &= 1+2t+t^2-2t^3-t^4+8t^5+20t^6+24t^7+18t^8+12t^9 \\
 &\quad +20t^{10}+44t^{11}+76t^{12}+94t^{13}+85t^{14}+58t^{15}+44t^{16} \\
 &\quad +58t^{17}+\dots+2t^{31}+t^{32}, \\
 D(T_{5,2}; t) &= (1-t)^2(1-t^2)^6(1-t^3)^8(1-t^4)^6(1-t^5)^4, \\
 N(T_{6,2}; t) &= 1-3t+4t^2-4t^3+4t^4+3t^5-6t^6+11t^7-12t^8+12t^9 \\
 &\quad +12t^{10}+t^{11}+55t^{12}-22t^{13}+82t^{14}+77t^{16}+119t^{17}+84t^{18} \\
 &\quad +234t^{19}+160t^{20}+227t^{21}+312t^{22}+207t^{23}+507t^{24}+301t^{25} \\
 &\quad +612t^{26}+469t^{27}+517t^{28}+593t^{29}+426t^{30}+593t^{31}+\dots-3t^{59}+t^{60}, \\
 D(T_{6,2}; t) &= (1-t)^7(1-t^2)^3(1-t^3)^6(1-t^4)^9(1-t^5)^7(1-t^6)^5.
 \end{aligned}$$

However, he did not compute an MSG for $C_{4,2}$. GL_2 acts (via standard representation) on the 2-dimensional space spanned by the matrices x and y . This action induces an action on $C_{4,2}$, which was investigated by Drensky and Sadikova [5]. They show that there is an MSG whose span is a semisimple graded GL_2 -submodule of $C_{4,2}$ and they determine the structure of this module. Its Poincaré polynomial is $2t + 3t^2 + 4t^3 + 6t^4 + 2t^5 + 4t^6 + 2t^7 + 4t^8 + 4t^9 + t^{10}$. Hence, an MSG consists of 32 polynomials. In their paper they do not list explicitly such a generating set. We have computed an MSG of $C_{4,2}$ independently:

Theorem 4.2. *The 17 traces mentioned in Theorem 4.1 together with the traces of the following 15 matrices (arranged by their degrees)*

$$\begin{aligned}
 &x^3y^2, y^3x^2; \quad x^2y^2xy, y^2x^2yx; \quad x^3y^2xy, y^3x^2yx; \\
 &x^3y^2x^2y, y^3x^2y^2x, x^3y^3xy, y^3x^3yx; \\
 &x^3yx^2yxy, x^2y^2xyx^2y, y^2x^2yxy^2x, y^3xy^2xyx; \quad x^3y^3x^2y^2
 \end{aligned}$$

form an MSG of $C_{4,2}$.

Proof. Our proof is computational. We start with the system of parameters from Teranishi’s theorem and consider it as the first approximation to the genuine generating set which we want to construct. Then we compare the Poincaré series of $C_{4,2}$ with that of its subalgebra generated by this HSOP. The difference of the former and the latter is a series with nonnegative integer coefficients. We find the first nonzero term, say ct^d . This means that we have to enlarge our incomplete generating set by additional c generators of degree d . We then select c words in x and y of length d , whose traces provide these additional generators. We repeat this procedure with the enlarged set of generators, and continue repeating it until we reach the space of invariants of degree 10. After adding the single additional generator in degree 10, we are certain that we have indeed found the full set of generators. This is a consequence of the well-known

fact [4, Part A, Theorem 6.1.6], originally due to Procesi [13], which establishes the connection between the Nagata–Higman theorem and the invariant theory of generic matrices. \square

Remark 4.3. The words $y^3x^2yx, y^3x^2y^2x, y^2x^2yx^2x, y^3xy^2xyx$ in Theorem 4.2 can be replaced with $xyx^2y^3, xy^2x^2y^3, xy^2xyx^2y^2, xyxy^2xy^3$, respectively.

The total number of generators listed in the theorem is 32. This number as well as the degrees of the generators are in agreement with the result of Drensky and Sadikova mentioned above.

Let us say that a set W of words in two noncommuting indeterminates x, y is a *test set* for $M_n(\mathbb{C})$ if it has the following property: Two matrices $a, b \in M_n(\mathbb{C})$ are unitarily similar iff $\text{tr}(w(a, a^*)) = \text{tr}(w(b, b^*))$ holds for all $w \in W$. Such a test set W is *minimal* if no proper subset of W is a test set. As a consequence of the above theorem, we obtain the following criterion.

Theorem 4.4. *The following 20 words form a test set for $M_4(\mathbb{C})$:*

$$x; \quad x^2, xy; \quad x^3, x^2y; \quad x^4, x^3y, x^2y^2, xyxy; \quad x^3y^2; \quad (x^2y)^2, x^2y^2xy, y^2x^2yx;$$

$$x^3y^2xy; \quad x^3y^2x^2y, x^3y^3xy, y^3x^3yx; \quad x^3yx^2yxy, x^2y^2xyx^2y; \quad x^3y^3x^2y^2.$$

Proof. First apply the above remark. Then among the 32 words $w(x, y)$ whose traces generate the algebra $C_{4,2}$, given in Theorem 4.2, there are 12 pairs $\{w_1, w_2\}$ such that, for any 4×4 complex matrix a , $\text{tr}(w_1(a, a^*))$ and $\text{tr}(w_2(a, a^*))$ are complex conjugates. For instance, $\{x, y\}$, $\{x^2, y^2\}$ and $\{x^3y^2, y^3x^2\}$ are such pairs. (For the nonpaired words, such as $w(x, y) = xyxy$, the trace of $w(a, a^*)$ is always real.) By dropping one of the words from each of these pairs, we obtain the test set in the theorem. \square

5. Generators of the algebra $C_{5,2}$

Let $M_n(0)$ be the subspace of M_n consisting of matrices of trace 0 and let $C_{n,2}(0) = K[M_n(0)^2]^{\text{GL}_n}$ be the corresponding algebra of GL_n -invariant polynomial functions on the direct product $M_n(0)^2 = M_n(0) \times M_n(0)$. Then one has an isomorphism of \mathbb{Z}^2 -graded algebras

$$C_{n,2} \cong K[u, v] \otimes C_{n,2}(0), \tag{5.1}$$

where $K[u, v]$ is the polynomial algebra in two variables u and v (see e.g. [13, Section 5]). Consequently, the problem of constructing an MSG of $C_{n,2}$ reduces to the same problem for $C_{n,2}(0)$. In the remainder of this section we shall assume that $x, y \in M_n(0)$.

In view of the above isomorphism, we have the following obvious relation between the Poincaré series of these algebras:

$$P(C_{n,2}; t) = \frac{P(C_{n,2}(0); t)}{(1-t)^2}.$$

Theorem 5.1. *The algebra $C_{5,2}(0)$ has an MSG \mathcal{P} consisting of 171 bihomogeneous polynomials. \mathcal{P} is the disjoint union of four subsets: $\mathcal{P}_s, \mathcal{P}_k, \mathcal{P}_d$ and \mathcal{P}'_d with cardinals 5, 4, 81 and 81, respectively. The polynomials $f(x, y) \in \mathcal{P}_s$ are symmetric (i.e., satisfy $f(y, x) = f(x, y)$),*

while those in \mathcal{P}_k are skew-symmetric. The polynomials in \mathcal{P}_d are neither symmetric nor skew-symmetric. There is a bijection $\mathcal{P}_d \rightarrow \mathcal{P}'_d$ given by $f(x, y) \rightarrow f(y, x)$. The sets \mathcal{P}_s , \mathcal{P}_k and \mathcal{P}_d are given below.

\mathcal{P}_s consists of the traces of the 5 words: $xy, x^2y^2, xyxy, x^3y^3, x^4y^4$. \mathcal{P}_k consists of the traces of the following 4 matrices:

$$\begin{aligned} &x^3y^3x^2y^2 - y^3x^3y^2x^2, \\ &x^2yxy^2xyxy - y^2xyx^2yxyx, \\ &x^3y^2xyxy^2xy - y^3x^2yxyx^2yx, \\ &x^4y^4x^3y^3 - y^4x^4y^3x^3. \end{aligned}$$

Finally, the set \mathcal{P}_d consists of the traces of the following 81 words:

$$\begin{aligned} &x^2; \quad x^3, x^2y; \quad x^4, x^3y; \quad x^5, x^4y, x^3y^2, x^2yxy; \\ &x^4y^2, (x^2y)^2, x^2y^2xy; \quad x^4y^3, x^4yxy, x^3y^2xy, x^3yxy^2; \\ &x^4y^2xy, x^4yx^2y, x^4yxy^2, x^3y^3xy, x^3y^2x^2y, x^2y^2xyxy; \\ &x^4y^2x^2y, x^4y^2xy^2, x^4yx^2y^2, x^3y^3x^2y, x^3y^2xyxy, x^3yxxy^2y, x^3yxyxy^2, x^2y^2xyx^2y; \\ &x^4y^4xy, x^4y^3xy^2, x^4y^2x^3y, x^4y^2x^2y^2, x^4y^2xyxy, x^4yx^2yxy, x^4yxy^2xy, \\ &\quad x^3y^3xyxy, x^3y^2x^2yxy, x^3yx^2yxy^2; \\ &x^4y^4x^2y, x^4y^4xy^2, x^4y^3x^3y, x^4y^3x^2y^2, x^4y^3xyxy, x^4y^2x^2yxy, x^4y^2xyxy^2, \\ &\quad x^4yx^3yxy, x^4yx^2y^2xy, x^4yx^2yxy^2, x^3y^2x^2y^2xy, x^3yxxyxy^2xy; \\ &x^4y^4x^3y, x^4y^4x^2y^2, x^4y^3x^3y^2, x^4y^3x^2yxy, x^4y^3xy^2xy, x^4y^2x^3yxy, x^4y^2x^2yx^2y, \\ &\quad x^4y^2xy^2x^2y, x^4y^2xyx^2y^2, x^4yx^3yx^2y, x^4yx^3yxy^2, x^4yxyxy^2xy, x^3y^3x^2yxy^2; \\ &x^4y^4x^3y^2, x^4y^4x^2yxy, x^4y^4xy^2xy, x^4y^3x^3yxy, x^4y^3x^2y^2xy, x^4y^3(x^2y)^2, \\ &\quad x^4y^2x^3yx^2y, x^4(y^2x^2)^2y, x^4y^2xyxy^2xy; \\ &x^4y^4x^3yxy, x^4y^4x^2y^2xy, x^4y^3x^3y^2xy, x^4y^3x^2yx^3y; \\ &x^4y^4x^3y^2xy, x^4y^3x^3y^2x^2y, x^4y^2x^3yx^2yxy. \end{aligned}$$

Proof. Shestakov and Zhukavets [15] have shown that any 2-generated associative algebra (nonunital and over a field of characteristic 0) which satisfies the identity $x^5 = 0$, also satisfies the identity $x_1x_2 \cdots x_{15} = 0$. Since we are working with only two generic matrices, by invoking a theorem of Procesi [13, Theorem 3.2], we conclude that the algebra $C_{5,2}$ (and $C_{5,2}(0)$) is generated by polynomials of degree at most 15.

Let \mathcal{A} denote the unital subalgebra of $C_{5,2}(0)$ generated by 171 polynomials in our set \mathcal{P} . We have verified, using a computer, that for each degree $d \leq 15$ the homogeneous component, \mathcal{A}_d , of \mathcal{A} of degree d has the dimension equal to the coefficient of t^d in the Poincaré series of the

algebra $C_{5,2}(0)$. As an additional check, we have computed the dimension of \mathcal{A}_{16} and verified that it is indeed equal to 17338. \square

Remark 5.2. In view of (5.1), an MSG of $C_{5,2}$ has cardinal 173.

Remark 5.3. One can modify \mathcal{P} by replacing each of the four generators $\text{tr}(w(x, y) - w(y, x)) \in \mathcal{P}_k$ with $\text{tr}(w(x, y))$. The only reason for our choice was to make \mathcal{P} stable (up to sign) under the involution which interchanges x and y .

6. Poincaré series for invariants of $GL_n \times SL_d$

M_n^d can be identified with $M_n \otimes K^d$. The action of GL_n on M_n^d corresponds to its action on this tensor product given by $a \cdot (x \otimes v) = axa^{-1} \otimes v$. We can now view $M_n \otimes K^d$ as a module for the direct product $GL_n \times GL_d$ by letting GL_d act on K^d by multiplication. Denote by $C_{n,d}^\#$ the subalgebra of $K[M_n^d]$ consisting of $GL_n \times SL_d$ -invariant functions. This is a subalgebra of $C_{n,d}$.

In this section we record (see Table 5) the Poincaré series of the algebras $C_{n,2}^\#$ for $n \leq 5$. We write these Poincaré series as rational functions in lowest terms

$$P(C_{n,2}^\#; t) = \frac{N(C_{n,2}^\#; t)}{D(C_{n,2}^\#; t)},$$

with usual normalization $N(C_{n,2}^\#; 0) = D(C_{n,2}^\#; 0) = 1$. The numerators are again palindromic.

Table 5
Numerators $N(C_{n,2}^\#; t)$ and denominators $D(C_{n,2}^\#; t)$

$N(C_{2,2}^\#; t) = 1,$	$D(C_{2,2}^\#; t) = (1 - t^4)^2,$	$N(C_{3,2}^\#; t) = 1 + 3t^8 + t^{10} + 3t^{12} + t^{20},$
$D(C_{3,2}^\#; t) = (1 - t^4)^3(1 - t^6)^3(1 - t^8),$		
$N(C_{4,2}^\#; t) = 1 + 10t^8 + 12t^{10} + 38t^{12} + 46t^{14} + 93t^{16} + 131t^{18} + 235t^{20}$		
$\quad + 299t^{22} + 473t^{24} + 560t^{26} + 714t^{28} + 761t^{30} + 876t^{32}$		
$\quad + 830t^{34} + 876t^{36} + \dots + 12t^{58} + 10t^{60} + t^{68},$		
$D(C_{4,2}^\#; t) = (1 - t^4)^3(1 - t^6)^4(1 - t^8)^4(1 - t^{10})^2(1 - t^{12}),$		
$N(C_{5,2}^\#; t) = 1 + t^2 + t^4 + t^6 + 14t^8 + 41t^{10} + 135t^{12} + 329t^{14} + 842t^{16}$		
$\quad + 1980t^{18} + 4677t^{20} + 10386t^{22} + 22654t^{24} + 47093t^{26}$		
$\quad + 94970t^{28} + 184182t^{30} + 346523t^{32} + 629769t^{34} + 1111589t^{36}$		
$\quad + 1902191t^{38} + 3165521t^{40} + 5120359t^{42} + 8066607t^{44}$		
$\quad + 12376177t^{46} + 18520117t^{48} + 27035364t^{50} + 38541637t^{52}$		
$\quad + 53673328t^{54} + 73078953t^{56} + 97307914t^{58} + 126802726t^{60}$		
$\quad + 161749890t^{62} + 202084191t^{64} + 247338162t^{66} + 296695937t^{68}$		
$\quad + 348874713t^{70} + 402270954t^{72} + 454898759t^{74} + 504632564t^{76}$		
$\quad + 549206297t^{78} + 586521387t^{80} + 614654835t^{82} + 632178319t^{84}$		
$\quad + 638112785t^{86} + 632178319t^{88} + \dots + 14t^{164} + t^{166} + t^{168} + t^{170} + t^{172},$		
$D(C_{5,2}^\#; t) = (1 + t^2 + t^4)(1 - t^4)^3(1 - t^6)^3(1 - t^8)^5(1 - t^{10})^5(1 - t^{12})^3$		
$\quad \cdot (1 - t^{14})^2(1 - t^{16})(1 - t^{18}).$		

Once again the Molien–Weyl formula was used. In this case the formula is more complicated:

$$P(C_{n,2}^\#; t) = \frac{1}{2\pi i} \int_{|y|=1} \frac{(1 - y^2)\psi(y)}{(1 - ty)^n (1 - \frac{t}{y})^n} \frac{dy}{y},$$

where

$$\begin{aligned} \psi(y) &= \frac{1}{(2\pi i)^{n-1}} \int_{|x_1|=1} \cdots \int_{|x_{n-1}|=1} \prod_{1 \leq k \leq r \leq n-1} \frac{1 - x_k x_{k+1} \cdots x_r}{\psi_{k,r}} \frac{dx_{n-1}}{x_{n-1}} \cdots \frac{dx_1}{x_1}, \\ \psi_{k,r} &= (1 - tyx_k x_{k+1} \cdots x_r)(1 - ty^{-1} x_k x_{k+1} \cdots x_r) \\ &\quad \cdot (1 - ty(x_k x_{k+1} \cdots x_r)^{-1})(1 - t(yx_k x_{k+1} \cdots x_r)^{-1}). \end{aligned}$$

Computations were easy for $n \leq 4$ but hard (lasting several days) for $n = 5$.

Since $C_{5,2}^\#$ is the algebra of SL_2 -invariants in $C_{5,2}$, the coefficient of t^{2k} in the Taylor series of $P(C_{5,2}^\#; t)$ must be the same as the coefficient of the Schur function $f_{k,k}$ in the formula for $P(C_{5,2}; t)$ displayed in Appendix A. It is easy to check that this is indeed the case for $k \leq 12$, which gives a further confirmation of our formula for $P(C_{5,2}^\#; t)$.

7. Poincaré series for invariants of $GL_n \times \Delta_{d-1}$

Restrict the action of $GL_n \times SL_d$ on M_n^d to $GL_n \times \Delta_{d-1}$, where Δ_{d-1} is the maximal torus of SL_d consisting of diagonal matrices. Denote by $C_{n,d}^\bullet$ the subalgebra of $K[M_n^d]$ consisting of $GL_n \times \Delta_{d-1}$ -invariant polynomial functions. This is a subalgebra of $C_{n,d}$. We record in Table 6 the Poincaré series of $C_{n,2}^\bullet$ for $n \leq 5$. Write these functions as

$$P(C_{n,2}^\bullet; t) = \frac{N(C_{n,2}^\bullet; t)}{D(C_{n,2}^\bullet; t)},$$

with usual normalization. The numerators are again palindromic.

The functions were computed by using the formula:

$$P(C_{n,2}^\bullet; t) = \frac{1}{2\pi i} \int_{|z|=1} P(C_{n,2}; tz^{-1}, tz) \frac{dz}{z},$$

where $P(C_{n,2}; s, t)$ is the bigraded Poincaré series of $C_{n,2}$ from Section 2.

8. Conjectures

On the basis of our computations, we propose four conjectures about the numerators $N(C_{n,2}; s, t)$ and $N(T_{n,2}; s, t)$ and the denominators $D(C_{n,2}; s, t)$ and $D(T_{n,2}; s, t)$. (See Section 2 for the definitions.)

Conjecture 8.1. *The denominators $D(C_{n,2}; s, t)$ and $D(T_{n,2}; s, t)$ can be written as products of binomials $1 - s^a t^b$, where a and b are nonnegative integers.*

Table 6

Numerators $N(C_{n,2}^\bullet; t)$ and denominators $D(C_{n,2}^\bullet; t)$

$$\begin{aligned}
 N(C_{1,2}^\bullet; t) &= 1, & D(C_{1,2}^\bullet; t) &= 1 - t^2, & N(C_{2,2}^\bullet; t) &= 1 + t^4, \\
 D(C_{2,2}^\bullet; t) &= (1 - t^2)^2(1 - t^4)^2, \\
 N(C_{3,2}^\bullet; t) &= 1 + 3t^4 + 6t^6 + 9t^8 + 6t^{10} + 12t^{12} + 6t^{14} + \dots + t^{24}, \\
 D(C_{3,2}^\bullet; t) &= (1 - t^2)^2(1 - t^4)^3(1 - t^6)^3(1 - t^8), \\
 N(C_{4,2}^\bullet; t) &= 1 + 4t^4 + 12t^6 + 36t^8 + 68t^{10} + 171t^{12} + 316t^{14} + 639t^{16} + 1096t^{18} \\
 &\quad + 1096t^{18} + 1849t^{20} + 2794t^{22} + 4151t^{24} + 5546t^{26} + 7229t^{28} + 8700t^{30} \\
 &\quad + 10085t^{32} + 10836t^{34} + 11270t^{36} + 10836t^{38} + \dots + 12t^{66} + 4t^{68} + t^{72}, \\
 D(C_{4,2}^\bullet; t) &= (1 - t^2)^2(1 - t^4)^3(1 - t^6)^4(1 - t^8)^4(1 - t^{10})^2(1 - t^{12}), \\
 N(C_{5,2}^\bullet; t) &= 1 + t^2 + 5t^4 + 20t^6 + 76t^8 + 227t^{10} + 692t^{12} + 1933t^{14} + 5307t^{16} + 13752t^{18} \\
 &\quad + 34304t^{20} + 81525t^{22} + 186346t^{24} + 408071t^{26} + 860437t^{28} + 1746504t^{30} \\
 &\quad + 3421732t^{32} + 6474866t^{34} + 11857662t^{36} + 21033945t^{38} + 36195856t^{40} \\
 &\quad + 60479854t^{42} + 98242554t^{44} + 155273212t^{46} + 239019423t^{48} + 358621723t^{50} \\
 &\quad + 524884888t^{52} + 749897456t^{54} + 1046516425t^{56} + 1427383948t^{58} \\
 &\quad + 1903851664t^{60} + 2484438301t^{62} + 3173436196t^{64} + 3969248353t^{66} \\
 &\quad + 4863282209t^{68} + 5838905156t^{70} + 6871421892t^{72} + 7928353846t^{74} \\
 &\quad + 8971036674t^{76} + 9956478001t^{78} + 10840418189t^{80} + 11580232480t^{82} \\
 &\quad + 12138549745t^{84} + 12485984964t^{86} + 12603960344t^{88} + 12485984964t^{90} \\
 &\quad + \dots + 20t^{170} + 5t^{172} + t^{174} + t^{176}, \\
 D(C_{5,2}^\bullet; t) &= (1 - t^2)(1 - t^4)^3(1 - t^6)^4(1 - t^8)^5(1 - t^{10})^5(1 - t^{12})^3 \\
 &\quad \cdot (1 - t^{14})^2(1 - t^{16})(1 - t^{18}).
 \end{aligned}$$

Conjecture 8.2. $N(C_{n,2}; 1, 1) = N(T_{n,2}; 1, 1) = 0$ for $n \geq 5$.

Conjecture 8.3. For all n ,

$$(1 - s)(1 - t)D(C_{n,2}; s, t) = (1 - s^n)(1 - t^n)D(T_{n,2}; s, t).$$

Conjecture 8.4. For all n ,

$$\gcd(N(C_{n,2}; t, t), D(C_{n,2}; t, t)) = \gcd(N(T_{n,2}; t, t), D(T_{n,2}; t, t)).$$

All four conjectures are true for $n \leq 6$. As all coefficients of $N(C_{n,2}^\#; t)$ and $N(C_{n,2}^\bullet; t)$ given in Sections 6 and 7 are nonnegative, we propose yet another conjecture.

Conjecture 8.5. For all n , $N(C_{n,2}^\#; t)$ and $N(C_{n,2}^\bullet; t)$ have nonnegative integer coefficients.

The following interesting problem may have some practical applications.

Problem 8.6. Construct minimal test sets for $M_4(\mathbb{C})$ and $M_5(\mathbb{C})$.

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Appendix A. Formanek’s formulae

Let d be any positive integer. Denote by μ a partition of a positive integer k and by χ^μ the corresponding irreducible complex character of the symmetric group S_k . Define the *length*, $l(\mu)$, of μ to be the number of parts of μ .

Let Λ_d denote the ring of symmetric polynomials in d variables, t_1, \dots, t_d . If μ has at most d parts, we denote by $f_{\mu,d} \in \Lambda_d$ the corresponding Schur function.

Define the Frobenius map Fr_d to be the additive homomorphism from the direct sum of the character rings of all S_k ’s to Λ_d by setting $\text{Fr}_d(\chi^\mu) = f_{\mu,d}$ for each partition μ of k having at most d parts and $\text{Fr}_d(\chi^\mu) = 0$ otherwise.

We can now state Formanek’s formulae for the algebras $C_{n,d}$ and $T_{n,d}$ (see [1]):

$$P(C_{n,d}; t_1, \dots, t_d) = \sum_{k \geq 0} \text{Fr}_d(\theta_n^{(k)})(t_1, \dots, t_d), \tag{A.1}$$

$$P(T_{n,d}; t_1, \dots, t_d) = \sum_{k \geq 0} \text{Fr}_d(\theta_n^{(k+1)} \downarrow S_k)(t_1, \dots, t_d), \tag{A.2}$$

where $\theta_n^{(k)}$ is a particular character of S_k . This character is defined by the formula

$$\theta_n^{(k)} = \sum_{\mu: l(\mu) \leq n} \chi^\mu \otimes \chi^\mu, \tag{A.3}$$

where \otimes is the usual tensor product of characters of S_k .

We are mainly interested in the case $d = 2$. In that case we set $s = t_1$, $t = t_2$, and $f_\mu = f_{\mu,2}$. If $\mu = (p, q)$ with $p \geq q \geq 0$, then

$$f_\mu = f_{p,q} = (st)^q (s^{p-q} + s^{p-q-1}t + \dots + st^{p-q-1} + t^{p-q}). \tag{A.4}$$

If $q = 0$ we shall write $f_p = f_{p,0}$.

Using GAP [10], we computed the first 26 terms of the series (A.1) when $(n, d) = (5, 2)$. The coefficients of the Schur functions $f_{k-p,p}$ are recorded in Table 7.

By substituting the expressions (A.4) for the Schur functions f_μ into this formula, one obtains an initial chunk of the bivariate Taylor series of $P(C_{5,2}; s, t)$ which agrees with the bivariate Taylor series of the rational function that we have computed by using the Molien–Weyl formula (see Theorem 2.2).

The k th summand in (A.1), when written as a linear combination of Schur functions, gives the decomposition of the character of the representation of GL_d on the k th homogeneous component of $C_{n,d}$.

Appendix B. Taylor expansion of $P(C_{n,2}; t)$

In this section we first tabulate the coefficients of the Taylor series of $P(C_{n,2}; t)$ for $n \leq 6$ including the terms of degree $k \leq 13$. Then we make a couple of observations, and we are lead to

Table 7
Coefficients of $f_{k-p,p}$ in the expansion of $P(C_{5,2}; s, t)$

k	$p = 0, 1, 2, \dots$
0	1;
1	1;
2	2;
3	3, 1;
4	5, 2, 3;
5	7, 5, 6;
6	10, 8, 15, 4;
7	13, 14, 25, 18;
8	18, 20, 44, 37, 24;
9	23, 30, 66, 76, 58;
10	30, 41, 101, 126, 136, 44;
11	37, 57, 142, 207, 246, 163;
12	47, 74, 200, 311, 431, 354, 171;
13	57, 98, 267, 458, 681, 700, 476;
14	70, 124, 357, 640, 1047, 1204, 1080, 388;
15	84, 157, 460, 884, 1517, 1973, 2024, 1277;
16	101, 194, 591, 1177, 2160, 3014, 3534, 2811, 1166;
17	119, 240, 740, 1550, 2957, 4470, 5682, 5427, 3419;
18	141, 290, 924, 1992, 3985, 6352, 8780, 9371, 7592, 2808;
19	164, 351, 1131, 2535, 5232, 8823, 12921, 15284, 14284, 8693;
20	192, 417, 1380, 3167, 6784, 11909, 18489, 23520, 24781, 18927, 7344;
21	221, 496, 1658, 3926, 8622, 15798, 25625, 34897, 40009, 35929, 21565;
22	255, 582, 1986, 4796, 10849, 20520, 34778, 49917, 61722, 61801, 46991, 17281;
23	291, 682, 2350, 5820, 13444, 26294, 46117, 69582, 91235, 100058, 87853, 51694;
24	333, 790, 2772, 6983, 16519, 33154, 60179, 94507, 130796, 153818, 150865, 111058, 41569;
25	377, 915, 3237, 8328, 20055, 41349, 77153, 125907, 182080, 227776, 242629, 207439, 120672;

some speculations concerning certain limits of the algebras $C_{n,d}$, which we are going to introduce now.

Define the \mathbf{Z}^d -graded algebra $C_{\infty,d}$ as the inverse limit of

$$C_{1,d} \leftarrow C_{2,d} \leftarrow C_{3,d} \leftarrow \dots$$

By adapting a definition of Formanek [8, p. 52], we refer to $C_{\infty,d}$ as the *pure free trace ring* on d generators. One can next take the direct limit of

$$C_{\infty,1} \rightarrow C_{\infty,2} \rightarrow C_{\infty,3} \rightarrow \dots$$

to obtain the \mathbf{Z}^∞ -graded algebra $C_{\infty,\infty}$, which is the pure free trace ring on countably many generators x_1, x_2, x_3, \dots . It follows from the Second Fundamental Theorem for invariants of $n \times n$ matrices (see e.g. [8, Theorem 50]) that $C_{\infty,\infty}$ is indeed isomorphic to the pure free trace ring as defined by Formanek.

Let us write $c_{n,2}(k)$ for the coefficients of the Poincaré series

$$P(C_{n,2}; t) = \sum_{k \geq 0} c_{n,2}(k)t^k$$

and display them in an infinite table (rows indexed by $n \geq 0$ and columns by $k \geq 0$). As we know the $P(C_{n,2}; t)$ for $n \leq 6$, we can fill the top portion of Table 8.

Table 8
The coefficients $c_{n,2}(k)$

1	0	0	0	0	0	0	0	0	0	0	0	0	...
1	2	3	4	5	6	7	8	9	10	11	12	13	
1	2	6	10	20	30	50	70	105	140	196	252	336	
1	2	6	14	29	56	107	186	320	530	851	1332	2051	
1	2	6	14	34	68	144	276	534	974	1774	3106	5410	
1	2	6	14	34	74	159	324	657	1286	2488	4702	8790	
1	2	6	14	34	74	166	342	716	1442	2898	5686	11122	
1	2	6	14	34	74	166	350	737	1512	3087	6194	12376	
1	2	6	14	34	74	166	350	746	1536	3168	6416	12982	
1	2	6	14	34	74	166	350	746	1546	3195	6508	13237	
1	2	6	14	34	74	166	350	746	1546	3206	6538	13340	
1	2	6	14	34	74	166	350	746	1546	3206	6550	13373	
1	2	6	14	34	74	166	350	746	1546	3206	6550	13386	
:													
:													

We observed from the top part that apparently each column stabilizes and

$$\lim_{n \rightarrow \infty} P(C_{n,2}; t) = \sum_{k \geq 0} c_{n,2}(n)t^n.$$

After making this observation, we looked up the diagonal sequence

$$\{c_{n,2}(n)\}_{n \geq 0} = 1, 2, 6, 14, 34, 74, 166, \dots$$

in the On-Line Encyclopedia of Integer Sequences [16] by entering only these 7 integers. It is registered there as the sequence A070933, and identified as the sequence of coefficients in the power series expansion of the infinite product

$$\prod_{k \geq 1} \frac{1}{1 - 2t^k}.$$

The first 30 terms of A070933 are listed in [16]. The above infinite product should be the Poincaré series of the algebra $C_{\infty,2}$. In the bigraded case it should be replaced with

$$\prod_{k \geq 1} \frac{1}{1 - s^k - t^k}.$$

More generally, we expect that the multigraded Poincaré series of $C_{\infty,d}$ and $C_{\infty,\infty}$ be given by

$$P(C_{\infty,d}; t_1, \dots, t_d) = \prod_{k \geq 1} \frac{1}{1 - t_1^k - \dots - t_d^k}$$

and

$$P(C_{\infty, \infty}; t_1, t_2, \dots) = \prod_{k \geq 1} \frac{1}{1 - p_k}, \tag{B.1}$$

respectively, where the p_k are the usual power sum symmetric functions:

$$p_k = t_1^k + t_2^k + \dots.$$

The latter formula is indeed valid. As explained in [7], it follows from the Procesi–Razmyslov theorem that

$$P(C_{n, \infty}; t) = \sum_{\mu; l(\mu) \leq n} \text{Fr}(\chi^\mu \otimes \chi^\mu).$$

The Frobenius map Fr is the additive map from the direct sum of the character rings of all S_k 's to the ring, Λ , of symmetric functions in infinitely many variables t_1, t_2, \dots . It is defined by setting, for all partitions μ , $\text{Fr}(\chi^\mu) = f_\mu \in \Lambda$, the Schur function corresponding to the partition μ . By letting $n \rightarrow \infty$, we obtain that

$$P(C_{\infty, \infty}; t) = \sum_{\mu} \text{Fr}(\chi^\mu \otimes \chi^\mu),$$

where the summation is now over all partitions μ . It remains to observe that the right-hand side of this formula and the one of (B.1) are equal, see Macdonald's classic [11, Chapter 1, Section 7, Example 9(a)].

Another interesting observation is that apparently the second differences

$$\alpha_k = c_{n,2}(n+k) - c_{n-1,2}(n+k) - c_{n-1,2}(n+k-1) + c_{n-2,2}(n+k-1)$$

are independent of n for $n \geq k$. The sequence

$$\{\alpha_k\}_{k \geq 0} = 1, 3, 11, 33, 98, 270, \dots \tag{B.2}$$

has not been recorded so far in [16]. By using the sequence A070933 and the hypothetical rules mentioned above, we extended the top portion of Table 8 with 6 additional rows. Subsequently, by using Formanek's formula, we enlarged the number of columns to 26, i.e., $0 \leq k \leq 25$. This made it possible to compute a few more terms of the sequence (B.2):

$$1, 3, 11, 33, 98, 270, 736, 1932, 5009, 12727, 31977, 79307, 194947, \dots$$

Appendix C. Taylor expansion of $P(T_{n,2}; t)$

Here we tabulate, for $n \leq 6$ and $k \leq 13$, the coefficients of t^k in the Taylor expansion of $P(T_{n,2}; t)$ and make some interesting observations. Let us define the coefficients $d_{n,2}(k)$ by

$$P(T_{n,2}; t) = \sum_{k \geq 0} d_{n,2}(k) t^k$$

Table 9
The coefficients $d_{n,2}(k)$

1	0	0	0	0	0	0	0	0	0	0	0	0	...
1	2	3	4	5	6	7	8	9	10	11	12	13	
1	4	11	24	46	80	130	200	295	420	581	784	1036	
1	4	14	38	93	204	419	806	1480	2600	4411	7244	11 579	
1	4	14	42	113	278	646	1418	2979	6018	11 752	22 256	41 030	
1	4	14	42	118	304	747	1748	3949	8620	18 296	37 818	76 398	
1	4	14	42	118	310	779	1876	4382	9948	22 057	47 850	101 844	
⋮													

and display them in an infinite table (rows indexed by $n \geq 0$ and columns by $k \geq 0$).

By taking the inverse limit of $T_{1,d} \leftarrow T_{2,d} \leftarrow T_{3,d} \leftarrow \dots$, one obtains the \mathbf{Z}^d -graded algebra $T_{\infty,d}$. By adapting a definition of Formanek [8, p. 52], we refer to $T_{\infty,d}$ as the *mixed free trace ring* on d generators. One can next take the direct limit of $T_{\infty,1} \rightarrow T_{\infty,2} \rightarrow T_{\infty,3} \rightarrow \dots$ to obtain the \mathbf{Z}^∞ -graded algebra $T_{\infty,\infty}$, which is the mixed free trace ring on countably many generators x_1, x_2, x_3, \dots .

There is a close relationship between Tables 8 and 9 from which we deduce that the formula $P(C_{\infty,2}; t) = (1 - 2t)P(T_{\infty,2}; t)$ apparently holds. By further heuristic reasoning, one obtains the hypothetical formulae

$$P(T_{\infty,d}; t_1, \dots, t_d) = \frac{1}{(1 - t_1 - \dots - t_d)^2} \prod_{k \geq 2} \frac{1}{1 - t_1^k - \dots - t_d^k}$$

and

$$P(T_{\infty,\infty}; t_1, t_2, \dots) = \frac{1}{(1 - p_1)^2} \prod_{k \geq 2} \frac{1}{1 - p_k}.$$

Similarly as in the previous appendix, the second differences of the coefficients $d_{n,2}(n + k)$ provide yet another sequence: 1, 6, 27, 103, 358, 1159, ...

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