Matrix Analysis (Lecture 4)

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Abstract

In the last lecture, we investigate properties of unitary matrices, introduce a special class of unitary matrices called Householder matrices, and leverage the norm preservation property of a Householder matrix (transformation) to deduce the important QR factorization. In this lecture, unitary similarity helps transform any given matrix into another matrix with a special form. Among all those special forms the Schur triangularization (or Schur form) is the most fundamentally useful result that embraces tremendous applications in elementary matrix analysis, either simplify calculations of some summary items of a matrix, or deduce other theorems that are related to eigenvalues of a matrix. In addition, a necessary and sufficient condition will be stated without proof.

1 Unitary Similarity (Page 94-98, Page 35)

Unitary similarity, like similarity, corresponds to a change of basis, but of a special type—it corresponds to a change from one *orthonormal* basis to another. We first summarize the definitions of unitary similarity as well as its real variation, real orthogonal similarity, into a formal statement.

Definition 1.1. *Let $A, B \in M_n$ be given.*

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(a) *We say that A is* unitarily similar *to B if there is a unitary $U \in M_n$ such that $A = U^*BU$. If U may be taken to be real (and hence is real orthogonal), then A is said to be* real orthogonally similar *to B.*

(b) *We say that A is* unitarily diagonalizable *if it is unitarily similar to a diagonal matrix; A is* real orthogonally diagonalizable *if it is real orthogonally similar to a diagonal matrix.*

Since unitary similarity is a special type of similarity, by Observation 1.2 in Lecture 2, we have an analogous observation.

Observation 1.2. *Unitary similarity is an equivalence relation.*

There are many necessities of two matrices to be unitarily similar. One obvious condition is that the Euclidean norms of their entries must be identical.

Theorem 1.3. *Let $U \in M_n$ and $V \in M_m$ be unitary, let $A = [a_{ij}] \in M_{n,m}$ and $B = [b_{ij}] \in M_{n,m}$, and suppose that $A = UBV$. Then*

$$\sum_{i,j=1}^{n,m} |b_{ij}|^2 = \sum_{i,j=1}^{n,m} |a_{i,j}|^2.$$

In particular, this identity is satisfied if $m = n$ and $V = U^*$, that is, if A is unitarily similar to B.

Proof. For any $A = [a_{ij}] \in M_{n,m}$, a direct computation shows that $\operatorname{tr} AA^* = \operatorname{tr} A^*A = \sum_{i,j=1}^{n,m} |a_{i,j}|^2$, so it suffices to prove that $\operatorname{tr} B^*B = \operatorname{tr} A^*A$. Using the fact that $\operatorname{tr} AB = \operatorname{tr} BA$, we compute that $\operatorname{tr} A^*A = \operatorname{tr}(UBV)^*(UBV) = \operatorname{tr}(V^*B^*U^*UBV) = \operatorname{tr}(V^*B^*BV) = \operatorname{tr}(B^*BVV^*) = \operatorname{tr} B^*B$. $\square$

Unitary similarity implies similarity but not conversely. Consider the matrices $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}$. They are similar via a matrix $S = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$, that is, $S^{-1}AS = B$. However, A and B are by no means unitarily similar to each other, since $\sum_{i,j=1}^{2} |a_{ij}|^2 = 6 \neq 14 = \sum_{i,j=1}^{2} |b_{i,j}|^2$. Therefore, the unitary similarity equivalence relation partitions M_n into *finer* equivalence classes than the similarity equivalence relation.

Theorem 1.3 provides a necessary but not sufficient condition for two given matrices to be unitarily similar. It can indeed be augmented with additional identities that collectively do provide necessary and sufficient conditions. A key role is played by the following simple notion. Let s, t be two given noncommuting variables. Any finite formal product of nonnegative powers of s and t

$$W(s,t) = s^{m_1}t^{n_1}s^{m_2}t^{n_2}\cdots s^{m_k}t^{n_k}, \quad m_1, n_1, ..., m_k, n_k \geq 0 \tag{1}$$

is called a *word in s and t*. The *length* of the word $W(s,t)$ is the nonnegative integer $m_1 + n_1 + m_2 + n_2 + \cdots + m_k + n_k$, i.e., the sum of all the exponents in the word.

If $A \in M_n$ is given, we define a *word in A and A^** as

$$W(A, A^*) = A^{m_1}(A^*)^{n_1}A^{m_2}(A^*)^{n_2}\cdots A^{m_k}(A^*)^{n_k}.$$

Since the powers of A and A^* need not commute, it may not be possible to simplify the expression of $W(A, A^*)$ by rearranging the terms in the product.

Suppose that A is unitarily similar to $B \in M_n$, that is, $A = UBU^*$ for some unitary $U \in M_n$. For any word $W(s,t)$ we have

$$\begin{aligned} W(A, A^*) &= (UBU^*)^{m_1}(UB^*U)^{n_1}\cdots(UBU^*)^{m_k}(UB^*U^*)^{n_k} \\ &= UB^{m_1}U^*U(B^*)^{n_1}U^*\cdots UB^{m_k}U^*U(B^*)^{n_k}U^* \\ &= UB^{m_1}(B^*)^{n_1}\cdots B^{m_k}(B^*)^{n_k}U^* \\ &= UW(B, B^*)U^* \end{aligned}$$

so $W(A, A^*)$ is unitarily similar to $W(B, B^*)$. Thus, $\operatorname{tr} W(A, A^*) = \operatorname{tr} W(B, B^*)$. Particularly, if we take the word $W(s,t) = st$, we obtain the identity in Theorem 1.3. If one considers all possible words $W(s,t)$, this will give rise to infinitely many necessary conditions for two matrices to be unitarily similar, which, surprisingly, are also sufficient.

Theorem 1.4 (Specht)**.** *Two matrices $A, B \in M_n$ are unitarily similar if and only if*

$$\operatorname{tr} W(A, A^*) = \operatorname{tr} W(B, B^*) \tag{2}$$

for every word $W(s,t)$ in two noncommuting variables.

Specht's theorem 1.4 can be used to show that two matrices are *not* unitarily similar by exhibiting a specific word that violates the identity (2). However, the applications of Specht's theorem 1.4 in determining whether two given matrices are unitarily similar are limited to some special situations.

Example 1.5. Let $A \in M_n$ and $B, C \in M_m$ be given. Then B and C are unitarily similar if and only if any one of the following conditions is satisfied:

(a) $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ and $\begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}$ are unitarily similar.

(b) $B \oplus \cdots \oplus B$ and $C \oplus \cdots \oplus C$ are unitarily similar if both direct sums contain the same number of direct summands.

(c) $A \oplus B \oplus \cdots \oplus B$ and $A \oplus C \oplus \cdots \oplus C$ are unitarily similar if both direct sums contain the same number of direct summands.

For (a), we know that $\operatorname{tr} W\Big(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \begin{bmatrix} A^* & 0 \\ 0 & B^* \end{bmatrix} \Big) = \operatorname{tr}\Big(\begin{bmatrix} W(A, A^*) & 0 \\ 0 & W(B, B^*) \end{bmatrix} \Big) = \operatorname{tr} W(A, A^*) + \operatorname{tr} W(B, B^*)$ and similarly, $\operatorname{tr} W\Big(\begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}, \begin{bmatrix} A^* & 0 \\ 0 & C^* \end{bmatrix} \Big) = \operatorname{tr} W(A, A^*) + \operatorname{tr} W(C, C^*)$. By Theorem 1.4, we conclude that

$$\begin{aligned} B \sim C &\iff \operatorname{tr} W(B, B^*) + \operatorname{tr} W(C, C^*) \\ &\iff \operatorname{tr} W\Big(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \begin{bmatrix} A^* & 0 \\ 0 & B^* \end{bmatrix} \Big) = \operatorname{tr} W\Big(\begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}, \begin{bmatrix} A^* & 0 \\ 0 & C^* \end{bmatrix} \Big) \\ &\iff \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \sim \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}. \end{aligned}$$

Likewise, for (b), we have $\operatorname{tr} W(B \oplus \cdots \oplus B, B^* \oplus \cdots \oplus B^*) = n \cdot \operatorname{tr} W(B, B^*)$ and $\operatorname{tr} W(C \oplus \cdots \oplus C, C^* \oplus \cdots \oplus C^*) = n \cdot \operatorname{tr} W(C, C^*)$.
For (c), we also have $\operatorname{tr} W(A \oplus B \oplus \cdots \oplus B, A^* \oplus B^* \oplus \cdots \oplus B^*) = \operatorname{tr} W(A, A^*) + n \cdot \operatorname{tr} W(B, B^*)$ and $\operatorname{tr} W(A \oplus C \oplus \cdots \oplus C, A^* \oplus C^* \oplus \cdots \oplus C^*) = \operatorname{tr} W(A, A^*) + n \cdot \operatorname{tr} W(C, C^*)$.
We can apply Theorem 1.4 and deduce (b) and (c) in a similar fashion.

In practice, it is useless to show that two given matrices *are* unitarily similar via Specht's theorem 1.4 because infinitely many conditions must be verified. Fortunately, a refinement of Specht's theorem 1.4 ensures that it suffices to check the trace identities (2) for only finitely many words, which can provide a practical criterion to assess unitary similarity of matrices of small size.

Theorem 1.6. *Let $A, B \in M_n$ be given.*

(a) A and B are unitarily similar if and only if the identity (2) is satisfied for every word $W(s,t)$ in two noncommuting variables whose length is at most

$$n\sqrt{\frac{2n^2}{n-1}+\frac{1}{4}}+\frac{n}{2}-2.$$

(b) If $n=2$, A and B are unitarily similar if and only if the identity (2) is satisfied for the 3 words $W(s,t)=s; s^2,$ and st.

(c) If $n=3$, A and B are unitarily similar if and only if the identity (2) is satisfied for the 7 words $W(s,t)=s, s^2, st; s^3, s^2t; s^2t^2;$ and s^2t^2st.

(d) If $n=4$, A and B are unitarily similar if and only if the identity (2) is satisfied for the 20 words in the following table:

s	s^2, st
s^3, s^2t	$s^4, s^3t, s^2t^2, stst$
s^3t^2	$s^2ts^2t, s^2t^2st, t^2s^2ts$
s^3t^2st	$s^3t^2s^2t, s^3t^3st, t^3s^3ts$
$s^3ts^2tst, s^2t^2sts^2t$	$s^3t^3s^2t^2$

Two *real* matrices are unitarily similar if and only if they are real orthogonally similar. Thus, the criteria in Theorem 1.4 and 1.6 are also necessary and sufficient for any two real matrices A and B to be real orthogonally similar.

For computational or theoretical reasons, it is often convenient to transform a given matrix by unitary similarity into another matrix with a special form. Here are two examples.

Example 1.7 (Unitary similarity to a matrix with equal diagonal entries)**.** Let $A=[a_{ij}]\in M_n$ be given. We claim that there is a unitary $U\in M_n$ such that all the main diagonal entries of $U^*AU=B=[b_{ij}]$ are equal; if A is real, then U may be taken to be real orthogonal. If this claim is true, then $\operatorname{tr} A=\operatorname{tr} B=nb_{11}$, so every main diagonal entry of B is equal to the average of the main diagonal entries of A.

We begin by considering the complex case and $n=2$. With loss of generality, we assume that $\operatorname{tr} A=0$, since we can replace $A\in M_2$ by $A-(\frac{1}{2}\operatorname{tr} A)I$. Then the two eigenvalues of A are $\pm\lambda$ for some $\lambda\in\mathbf{C}$. We wish to determine a unit vector u such that $u^*Au=0$. If $\lambda=0$, let u be any

unit vector such that $Au = 0$. If $\lambda \neq 0$, let w and z be any unit eigenvectors associated with the distinct eigenvalues $\pm\lambda$. Let $x(\theta) = e^{i\theta}w + z$, which is nonzero for all $\theta \in \mathbf{R}$ since w and z are linearly independent. Compute $x(\theta)^* Ax(\theta) = \lambda(e^{i\theta}w + z)^*(e^{i\theta}w - z) = 2i\lambda\mathrm{Im}(e^{i\theta}z^*w)$. If $z^*w = e^{i\phi}|z^*w|$, then $x(-\phi)^*Ax(-\phi) = 0$. Take $u = \frac{x(-\phi)}{||x(-\phi)||_2}$.
Now let $v \in \mathbf{C}^2$ be any unit vector that is orthogonal to u and let $U = [u\ v]$. Then U is unitary and $U^*AU)_{11} = u^*Au = 0$. But $\mathrm{tr}(U^*AU) = \mathrm{tr}(AUU^*) = \mathrm{tr}(A)$, so $(U^*AU)_{22} = 0$ as well.

Now suppose that $n = 2$ and A is real. If the diagonal entries of $A = [a_{ij}]$ are not equal, consider the plane rotation matrix $U_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. A calculation reveals that the diagonal entries of $U_\theta A U_\theta^T$ are equal if $(\cos^2\theta - \sin^2\theta)(a_{11} - a_{22}) = (2\sin\theta\cos\theta)(a_{12} + a_{21})$, so equal diagonal entries are achieved if $\theta \in (0, \frac{\pi}{2})$ is chosen so that $\cot 2\theta = \frac{a_{12}+a_{21}}{a_{11}-a_{22}}$.

We have now shown that any 2-by-2 complex matrix A is unitarily similar to a matrix with both diagonal entries equal to the average of the diagonal entries of A; if A is real, the similarity may be taken to be real orthogonal.

Now suppose that $n > 2$ and define $f(A) = \max\{|a_{ii} - a_{jj}| : i, j = 1, 2, ..., n\}$. If $f(A) > 0$, let $A_2 = \begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix}$ for a pair of indices i, j for which $f(A) = |a_{ii} - a_{jj}|$ (there could be several pairs of indices for which this maximum positive separation is attained; choose any one of them). Preceding arguments guarantee that there is a unitary $U_2 \in M_2$ (or real orthogonal if A is real) such that $U_2^* A_2 U_2$ has both main diagonal entries equal to $\frac{1}{2}(a_{ii}+a_{jj})$. Construct $U(i,j) \in M_n$ from U_2 in the same way that a plane rotation matrix $U(\theta; i, j)$ was constructed from a 2-by-2 plane rotation in Example 2.18 of Lecture 3. The unitary similarity $U(i,j)^*AU(i,j)$ affects only entries in rows and columns i and j, so it leaves unchanged every main diagonal entry of A except the entries in positions i and j, which it replaces with the average $\frac{1}{2}(a_{ii} + a_{jj})$. For any $k \neq i, j$ the triangle inequality ensures that

$$\begin{aligned} |a_{kk} - \frac{1}{2}(a_{ii} + a_{jj})| &= |\frac{1}{2}(a_{kk} - a_{ii}) + \frac{1}{2}(a_{kk} - a_{jj})| \\ &\leq \frac{1}{2}|a_{ii} - a_{kk}| + \frac{1}{2}|a_{kk} - a_{jj}| \\ &\leq \frac{1}{2}f(A) + \frac{1}{2}f(A) = f(A) \end{aligned}$$

with equality only if the scalars $a_{kk}-a_{ii}$ and $a_{kk}-a_{jj}$ both lies on the same ray in the complex plane and $|a_{kk}-a_{ii}| = |a_{kk}-a_{jj}|$. These two conditions imply that $a_{ii} = a_{jj}$, so it follows that $|a_{kk} - \frac{1}{2}(a_{ii}+a_{jj})| < f(A)$ for all $k \neq i, j$. Thus, the unitary similarity we have just constructed reduces by one the finitely many pairs of indices k, ℓ for which $f(A) = |a_{kk} - a_{\ell\ell}|$. Repeat the construction, if necessary, to deal with any such remaining pairs and achieve a unitary U (real if A is real) such that $f(U^*AU) < f(A)$.

Finally, consider the compact set $R(A) = \{U^*AU : U \in M_n \text{ is unitary}\}$ (Every entry of U^*AU is bounded by $||A||$; $\forall B_k \in R(A), k = 1, 2, ..., \lim_{k\to\infty} B_k = \lim_{k\to\infty} U_k^* A U_k = U^*AU = B \in R(A)$, where the limits are taken entry-wisely, so $R(A)$ is closed). Since f is a continuous nonnegative-valued function on $R(A)$, it achieves its minimum value there, that is, there is some $B \in R(A)$ such that $f(A) \geq f(B) \geq 0$ for all $A \in R(A)$. If $f(B) > 0$, we have seen that there is a unitary U (real if A is real) such that $f(B) > f(U^*BU)$. This contradiction shows that $f(B) = 0$, so all the diagonal entries of B are equal.

Example 1.8 (Unitary similarity to an upper Hessenberg matrix)**.** A matrix $B = [b_{ij}] \in M_n$ is said to be in *upper Hessenberg form* or to be an *upper Hessenberg matrix* if $b_{ij} = 0$ for all $i > j+1$:

$$\begin{bmatrix} b_{11} & & & & \bigstar \\ b_{21} & b_{22} & & & \\ & b_{32} & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & b_{n-1,n} & b_{nn} \end{bmatrix}$$

Let $A = [a_{ij}] \in M_n$ be given. The following construction shows that A is unitarily similar to an upper Hessenberg matrix with nonnegative entries in its first subdiagonal.

Let a_1 be the first column of A, partitioned as $a_1^T = [a_{11}\ \xi^T]$ with $\xi \in \mathbf{C}^{n-1}$. Let $U_1 = I_{n-1}$ if $\xi = 0$; otherwise, use Theorem 2.21 in Lecture 3 to construct $U_1 = U(||\xi||_2 e_1, e_1) \in M_{n-1}$, a unitary matrix that takes ξ into a positive multiple of e_1. Form the unitary matrix $V_1 = I_1 \oplus U_1$ and observe that the first column of $V_1 A$ is the vector $[a_{11}\ ||\xi||_2\ \ 0]^T$. Moreover,

$$B_1 = V_1 A V_1^* = \begin{bmatrix} a_{11} & \bigstar \\ \begin{bmatrix} ||\xi||_2 \\ 0 \end{bmatrix} & A_2 \end{bmatrix}, \quad A_2 \in M_{n-1},$$

where B_1 is also unitarily similar to A. Similarly, let $a_1^{(2)}$ be the first column of A_2, partitioned as $a_1^{(2)} = [a_{11}^{(2)}\ \eta^T]$ with $\eta \in \mathbf{C}^{n-2}$. Let $U_2 = I_{n-2}$ if $\eta = 0$; otherwise, use Theorem 2.21 in Lecture 3 again to form $U_2 = U(||\eta||_2 e_1, \eta) \in M_{n-2}$. Let $V_2 = I_2 \oplus U_2$ and let $B_2 = V_2 B_1 V_2^*$; the first column of B_1 is undisturbed, while the first column of A_2 is taken to be a vector whose entries below the second are all zero and whose second entry is nonnegative.

After $n-2$ of these reductions, we obtain an upper Hessenberg matrix B_{n-2} that is unitarily similar to A and has nonnegative subdiagonal entries except perhaps for the entry in position $(n, n-1)$; a final unitary similarity via $I_{n-1} \oplus [e^{i\theta}]$ may be necessary to rotate it to be nonnegative.

2 Unitary and real orthogonal triangularizations (Page 101-106, Page 31-32, 1.3.P33)

We have transformed a given matrix by unitary similarity into another matrix with equal diagonal entries or an upper Hessenberg matrix with nonnegative entries in its first subdiagonal. Now we are supposed to transform any square matrix A via unitary similarity into a triangular matrix whose diagonal entries are the eigenvalues of A, in any prescribed order. This triangularization technique, though rather simple in its proof, brings about great influences on other ramifications of matrix analysis, which we will discuss in detail in subsequent lectures.

Theorem 2.1 (Schur form; Schur triangularization). *Let $A \in M_n$ have eigenvalues $\lambda_1, ..., \lambda_n$ in any prescribed order and let $x \in \mathbf{C}^n$ be a unit vector such that $Ax = \lambda_1 x$.*

(a) There is a unitary $U = [x\ u_2 \cdots u_n] \in M_n$ such that $U^*AU = T = [t_{ij}]$ is upper triangular with diagonal entries $t_{ii} = \lambda_i, i = 1, ..., n$.

(b) If $A \in M_n(\mathbf{R})$ has only real eigenvalues, then x may be chosen to be real and there is a real orthogonal $Q = [x\ q_2 \cdots q_n] \in M_n(\mathbf{R})$ such that $Q^TAQ = T = [t_{ij}]$ is upper triangular with diagonal entries $t_{ii} = \lambda_i, i = 1, ..., n$.

Proof. Let x be a normalized eigenvector of A associated with the eigenvalue λ_1, that is, $x^*x = 1$ and $Ax = \lambda_1 x$. Let $U_1 = [x\ u_2 \cdots u_n]$ be any unitary matrix whose first column is x. Here we give two different constructions of U_1:

- one may take $U_1 = U(x, e_1)$ as in Theorem 2.21 of Lecture 3
- Write $x = [x_1\ y^T]^T$, where $x_1 \in \mathbf{C}$ and $y \in \mathbf{C}^{n-1}$. Choose $\theta \in \mathbf{R}$ such that $e^{i\theta}x_1 \geq 0$ and define $z = e^{i\theta}x = [z_1\ \zeta^T]^T$, where $z_1 \in \mathbf{R}$ is nonnegative and $\zeta \in \mathbf{C}^{n-1}$. Consider the Hermitian matrix

$$V_x = \left[\begin{array}{c|c} z_i & \zeta^* \\ \hline \zeta & -I + \frac{1}{1+z_1}\zeta\zeta^* \end{array}\right].$$

  $U_1 = e^{-i\theta}V_x$ is a unitary matrix whose first column is the given vector x.

Then

$$\begin{aligned} U_1^* A U_1 &= U_1^*[Ax\ Au_2 \cdots Au_n] = U_1^*[\lambda_1 x\ Au_2 \cdots Au_n] \\ &= \begin{bmatrix} x^* \\ u_2^* \\ \vdots \\ u_n^* \end{bmatrix} [\lambda_1 x\ Au_2 \cdots Au_n] \\ &= \begin{bmatrix} \lambda_1 x^* x & x^* A u_2 & \cdots & x^* A u_n \\ \lambda_1 u_2^* x & & & \\ \vdots & & A_1 & \\ \lambda_1 u_n^* x & & & \end{bmatrix} = \begin{bmatrix} \lambda_1 & \bigstar \\ 0 & A_1 \end{bmatrix} \end{aligned}$$

because the columns of U_1 are orthonormal. The eigenvalues of the submatrix $A_1 = [u_i^* A u_j]_{i,j=2}^n \in M_{n-1}$ are $\lambda_2, ..., \lambda_n$. If $n = 2$, we have achieved the desired unitary triangularization. If not, let $\xi \in \mathbf{C}^{n-1}$ be a unit eigenvector of A_1 associated with λ_2, and perform the preceding reduction on A_1. If $U_2 \in M_{n-1}$ is any unitary matrix whose first column is ξ, then we have seen that

$$U_2^* A_1 U_2 = \begin{bmatrix} \lambda_2 & \bigstar \\ 0 & A_2 \end{bmatrix}$$

Let $V_2 = I_1 \oplus U_2$ and compute the unitary similarity

$$(U_1V_2)^* A U_1 V_2 = V_2^* U_1^* A U_1 V_2 = \begin{bmatrix} \lambda_1 & \bigstar & \bigstar \\ 0 & \lambda_2 & \bigstar \\ 0 & 0 & A_2 \end{bmatrix}$$

Continue this reduction to produce unitary matrices $U_i \in M_{n-i+1}, i = 1, ..., n-1$ and unitary matrices $V_i \in M_n$ and unitary matrices $V_i = (I_{i-1} \oplus U_i) \in$

$M_n, i = 2, ..., n-1$. The matrix $U = U_1 V_2 V_3 \cdots V_{n-1}$ is unitary and U^*AU is upper triangular.

If all the eigenvalues of $A \in M_n(\mathbf{R})$ are real, then all of the eigenvectors and unitary matrices in the preceding algorithm can be chosen to be real. □

Remark. Applying Theorem 2.1 to A^T, we conclude that there is a unitary matrix $U \in M_n$ such that U^*A^TU is upper triangular. Let $V = \bar{U}$, which is also unitary, and thus $V^*AV = U^TA\bar{U} = (U^*A^TU)^T$ is lower triangular.

Example 2.2. If the eigenvalues of A are reordered and the corresponding upper triangularization in Theorem 2.1 is performed, the entries of T above the main diagonal can be different. Consider

$$T_1 = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}, T_2 = \begin{bmatrix} 2 & -1 & 3\sqrt{2} \\ 0 & 1 & \sqrt{2} \\ 0 & 0 & 3 \end{bmatrix}, U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

Then U is unitary and $T_2 = UT_1U^*$.

Corollary 2.3 (Schur's inequality; defect from normality)**.** *If $A = [a_{ij}] \in M_n$ has eigenvalues $\lambda_1, ..., \lambda_n$ and is unitarily similar to an upper triangular matrix $T = [t_{ij}] \in M_n$, the diagonal entries of T are the eigenvalues of A in some order. Then*

$$\sum_{i=1}^{n} |\lambda_i|^2 = \sum_{i,j=1}^{n} |a_{ij}|^2 - \sum_{i<j} |t_{ij}|^2 \leq \sum_{i,j=1}^{n} |a_{ij}|^2 = \mathrm{tr}(AA^*) \tag{3}$$

with equality if and only if T is diagonal.

Proof. By Theorem 2.1, there exists a unitary matrix $U \in M_n$ such that $T_1 = U^*AU$ is upper triangular with diagonal entries $\lambda_1, ..., \lambda_n$. Since T is unitarily similar to A, it is also unitarily similar to T_1. Using the fact that untarily similar matrices have the same characteristic polynomial, we conclude that the diagonal entries of T are $\lambda_1, ..., \lambda_n$ in some order.

Applying Theorem 1.3 to A and T, we know that $\sum_{i=1}^{n} |\lambda_i|^2 + \sum_{i<j} |t_{ij}|^2 = \sum_{i,j=1}^{n} |a_{ij}|^2$. Therefore,

$$\sum_{i=1}^{n} |\lambda_i|^2 = \sum_{i,j=1}^{n} |a_{ij}|^2 - \sum_{i<j} |t_{ij}|^2 \leq \sum_{i,j=1}^{n} |a_{ij}|^2 = \mathrm{tr}(AA^*)$$

with equality if and only if $t_{ij} = 0, 1 \leq i < j \leq n$, that is, T is diagonal. □

Lemma 1.20 in Lecture 2 tells us that there is a common eigenvector for every matrix in a commuting family. This result serves as the fundamental building block for a useful extension of Theorem 2.1: A commuting family of complex matrices can be reduced simultaneously to upper triangular form by a single unitary similarity.

Theorem 2.4. *Let $\mathcal{F} \subseteq M_n$ be a nonempty commuting family. There is a unitary $U \in M_n$ such that U^*AU is upper triangular for every $A \in \mathcal{F}$.*

Proof. Return to the proof of Theorem 2.1. Exploiting Lemma 1.20 in Lecture 2 at each step of the proof in which a choice of an eigenvector (and unitary matrix) is made, choose a unit eigenvector that is common to every $A \in \mathcal{F}$ and construct a unitary matrix that has this common eigenvector as its first column; it deflates (via unitary similarity) every matrix in $\mathcal{F}$ in the same way. Similarity preserves commutativity, and a partitioned multiplication calculation reveals that, if two matrices of the form $\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ and $\begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}$ commute, then A_{22} and B_{22} commute also. We conclude that all ingredients in the U of Theorem 2.1 may be chosen in the same way for all members of a commuting family. $\square$

Remark. In Theorem 2.1 we may specify the main diagonal of T (that is, we may specify in advance the order where the eigenvalues of A appear as the deflation progresses), but Theorem 2.4 makes no such claim. At each stage of the deflation, the common eigenvector used is associated with *some* eigenvalue of *each* matrix in $\mathcal{F}$, but we may not be able to specify which one. We must take the eigenvalues as they come, according to the common eigenvectors guaranteed by Lemma 1.20 in Lecture 2.

In Theorem 2.1, if a real matrix A has any non-real eigenvalues, there is no hope of reducing it to upper triangular form T by a *real* similarity because some main diagonal entries of T (eigenvalues of A) would be non-real. However, we can make a compromise and reduce A to a real upper quasitriangular form by a real orthogonal similarity; conjugate pairs of non-real eigenvalues are associated with 2-by-2 blocks. To prove this result, we first define the notion of upper quasitriangular matrices and exploit a lemma for the real Schur form.

Definition 2.5. *A matrix $A \in M_n$ of the form*

$$A = \begin{bmatrix} A_{11} & \bigstar & \bigstar \\ & \ddots & \bigstar \\ \mathbf{0} & & A_{kk} \end{bmatrix}$$

where $A_{ii} \in M_{n_i}, i = 1, ..., k, \sum_{i=1}^{k} n_i = n$, and all blocks below the block diagonal are zero, is block upper triangular*; it is strictly block triangular if, in addition, all the diagonal blocks are zero blocks.*

A block upper triangular matrix where all the diagonal blocks are 1*-by-*1 *or* 2*-by-*2 *is said to be* upper quasitriangular*.*

Lemma 2.6. *Suppose that $A \in M_n(\mathbf{R})$ has a non-real eigenvalue λ and write $\lambda = a + ib$ with $a, b \in \mathbf{R}$ and $b > 0$. Let x be an eigenvector of A associated with λ and write $x = u + iv$ with $u, v \in \mathbf{R}^n$.*

(a) $\bar{\lambda}, \bar{x}$ is an eigenpair of A.

(b) u and v are linearly independent.

(c) There exists a nonsingular matrix $S \in M_n(\mathbf{R})$ such that $S^{-1}AS = \begin{bmatrix} B & \bigstar \\ 0 & A_1 \end{bmatrix}$, where $A_1 \in M_{n-2}(\mathbf{R})$ and $B = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. Thus, a real square matrix with a non-real eigenvalue λ is real similar to a 2-by-2 block upper triangular matrix whose upper left block reveals the real and imaginary parts of λ.

(d) The multiplicity of each of λ and $\bar{\lambda}$ as an eigenvalue of A_1 is 1 less than its multiplicity as an eigenvalue of A.

Proof. (a) Since $A \in M_n(\mathbf{R})$, we take complex conjugate on both sides of $Ax = \lambda x$ and obtain that $\bar{A}\bar{x} = A\bar{x} = \bar{\lambda}\bar{x}$.
(b) x and $\bar{x}$ are eigenvectors of A associated with different eigenvalues, since $\lambda \neq \bar{\lambda}$. By Lemma 1.11 in Lecture 2, x and $\bar{x}$ are linearly independent. Suppose that $\alpha u + \beta v = 0$ with $\alpha, \beta \in \mathbf{R}$. Substitute u, v with $\frac{x+\bar{x}}{2}, \frac{x-\bar{x}}{2i}$, respectively. Then we have $\alpha(\frac{x+\bar{x}}{2}) + \beta(\frac{x-\bar{x}}{2i}) = (\frac{\alpha-i\beta}{2})x + (\frac{\alpha+i\beta}{2})\bar{x} = 0$, so $\alpha - i\beta = \alpha + i\beta = 0$. Therefore, $\alpha = \beta = 0$ and u, v are linearly independent.
(c) Since u, v are linearly independent, we can choose any $S_1 = [s_3 \cdots s_n]$ such that $\{u, v, s_3, ..., s_n\}$ is a basis for $\mathbf{R}^n$; $S = [u\ v\ S_1]$ has linearly independent columns, so it is nonsingular. We know that $Au + iAv = Ax = \lambda x =$

$(a+bi)(u+iv) = (au-bv)+i(bu+av)$. Equating the real and imaginary parts of the identity shows that $A[u\ v] = [u\ v]B$, where $B = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. Thus, by $S^{-1}[u\ v] = \begin{bmatrix} I_2 \\ 0 \end{bmatrix}$, we obtain that

$$S^{-1}AS = S^{-1}\begin{bmatrix} A[u\ v] & AS_1 \end{bmatrix} = S^{-1}\begin{bmatrix} [u\ v]B & AS_1 \end{bmatrix} = \begin{bmatrix} B & \bigstar \\ 0 & A_1 \end{bmatrix}$$

where $A_1 \in M_n(\mathbf{R})$, since all matrices in the preceding identity are real.
(d) By Theorem 1.3 in Lecture 2, we know that similarity preserves the characteristic polynomial, so $p_A(t) = p_B(t)p_{A_1}(t) = (t-\lambda)(t-\bar{\lambda})p_{A_1}(t)$. $\square$

Theorem 2.7 (real Schur form). *Let $A \in M_n(\mathbf{R})$ be given.*

(a) There is a real nonsingular $S \in M_n(\mathbf{R})$ such that $S^{-1}AS$ is a real upper quasitriangular matrix

$$\begin{bmatrix} A_1 & & & \bigstar \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_m \end{bmatrix}, \quad \textit{each } A_i \textit{ is } 1\textit{-by-}1 \textit{ or } 2\textit{-by-}2 \tag{4}$$

with the following properties: (i) its 1-by-1 diagonal blocks display the real eigenvalues of A; (ii) each of its 2-by-2 diagonal blocks has a special form that displays a conjugate pair of non-real eigenvalues of A:

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}, a, b \in \mathbf{R}, b > 0, \quad \textit{and } a \pm ib \textit{ are eigenvalues of } A \tag{5}$$

(iii) its diagonal blocks are completely determined by the eigenvalues of A; they may appear in any prescribed order.

(b) There is a real orthogonal $Q \in M_n(\mathbf{R})$ such that Q^TAQ is a real upper quasitriangular matrix with the following properties: (i) its 1-by-1 diagonal blocks display the real eigenvalues of A; (ii) each of its 2-by-2 diagonal blocks has a conjugate pair of non-real eigenvalues (but no special form); (iii) the ordering of its diagonal blocks may be prescribed in the following sense: If the real eigenvalues and conjugate pairs of non-real eigenvalues of A are listed in a prescribed order, then the real eigenvalues and conjugate pairs of non-real eigenvalues of the respective diagonal blocks $A_1, ..., A_m$ of Q^TAQ are in the same order.

Proof. (a) Given any *real* eigenpair, the proof of Theorem 2.1 shows how to deflate A by a real orthogonal similarity; the deflation produces a real 1-by-1 diagonal block and a deflated matrix of the form $\begin{bmatrix} \lambda & * \\ 0 & A_1 \end{bmatrix}$. On the other hand, given an eigenpair whose eigenvalue is not real, we apply the previous Lemma 2.6 to deflate A via a real similarity; the deflation produces a real 2-by-2 diagonal block B of the special form (5) and a deflated matrix of the form $\begin{bmatrix} B & * \\ 0 & A_1 \end{bmatrix}$. Only finitely many deflations are needed to construct a nonsingular S such that $S^{-1}AS$ has the asserted upper quasitriangular form. We can control the order in which the diagonal blocks appear by choosing each deflation step, a particular eigenvalue and corresponding eigenvector.
(b) Suppose that an ordering of the real and conjugate non-real pairs of eigenvalues of A has been given, and let S be a nonsingular real matrix such that $S^{-1}AS$ has the form (5) with diagonal blocks in the prescribed order. Use QR factorization (Theorem 2.23 in Lecture 3) to factor S as $S = QR$, where Q is real orthogonal and R is real and upper triangular. Partition $R = [R_{ij}]$ conformally to the form (4) and compute $S^{-1}AS = R^{-1}Q^TAQR$, so

$$
\begin{aligned}
Q^TAQ &= R \begin{bmatrix} A_1 & & & \bigstar \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_m \end{bmatrix} R^{-1} \\
&= \begin{bmatrix} R_{11}A_1R_{11}^{-1} & & & \bigstar \\ & R_{22}A_2R_{22}^{-1} & & \\ & & \ddots & \\ 0 & & & R_{mm}A_mR_{mm}^{-1} \end{bmatrix}
\end{aligned}
$$

is upper quasitriangular, its 1-by-1 diagonal blocks are identical to those of the form (4), and its 2-by-2 diagonal blocks are similar to the corresponding blocks of the form (4). $\square$

As how we generalize Theorem 2.1 to a commuting family, there is also a commuting families version of the preceding theorem: A commuting family of real matrices may be reduced simultaneously to a common upper quasitriangular form by a single real or real orthogonal similarity. It is convenient to describe the partitioned structure of the form (4) by saying that it is partitioned conformally to a given quasidiagonal matrix $D = J_{n_1} \oplus \cdots \oplus J_{n_m} \in M_n$, where J_k denotes the k-by-k all-ones matrix and each n_j is either 1 or 2.

Theorem 2.8. *Let $\mathcal{F} \subseteq M_n(\mathbf{R})$ be a nonempty commuting family.*

(a) There is a nonsingular $S \in M_n(\mathbf{R})$ and a quasidiagonal $D = J_{n_1} \oplus \cdots \oplus J_{n_m} \in M_n$ such that: (i) for each $A \in \mathcal{F}$, $S^{-1}AS$ is a real upper quasitriangular matrix of the form

$$\begin{bmatrix} A_1(A) & & & \bigstar \\ & A_2(A) & & \\ & & \ddots & \\ 0 & & & A_m(A) \end{bmatrix} \tag{6}$$

that is partitioned conformally to D; (ii) if $n_j = 2$, then for each $A \in \mathcal{F}$ we have

$$A_j(A) = \begin{bmatrix} a_j(A) & b_j(A) \\ -b_j(A) & a_j(A) \end{bmatrix} \in M_2(\mathbf{R}) \tag{7}$$

and $a_j(A) \pm i b_j(A)$ are eigenvalues of A; and (iii) for each $j \in \{1, ..., m\}$ such that $n_j = 2$, there is some $A \in \mathcal{F}$ such that $b_j(A) \neq 0$. If every matrix in $\mathcal{F}$ has only real eigenvalues, then $S^{-1}AS$ is upper triangular for every $A \in \mathcal{F}$.

(b) There is a real orthogonal $Q \in M_n(\mathbf{R})$ and a quasidiagonal $D = J_{n_1} \oplus \cdots \oplus J_{n_m} \in M_n$ such that (i) for each $A \in \mathcal{F}$, $Q^T AQ$ is a real upper quasitriangular matrix of the form (6) that is partitioned conformally to D, and (ii) for each $j \in \{1, ..., m\}$ such that $n_j = 2$, there is some $A \in \mathcal{F}$ such that $A_j(A)$ has a conjugate pair of non-real eigenvalues. If every matrix in $\mathcal{F}$ has only real eigenvalues, then $Q^T AQ$ is upper triangular for every $A \in \mathcal{F}$.

Proof. (a) Following the inductive pattern of the proof of Theorem 2.4, it suffices to construct a nonsingular real matrix that deflates (via similarity) each matrix in $\mathcal{F}$ in the same way. Use Lemma 1.2 in Lecture 2 to choose a common unit eigenvector $x \in \mathbf{C}^n$ of every $A \in \mathcal{F}$. Write $x = u + iv$, where $u, v \in \mathbf{R}^n$.

There are two possibilities, the first of which is (i) $\{u, v\}$ is linearly dependent. In this event, there is a real unit vector $w \in \mathbf{R}^n$ and real scalars α, β, not both zero, such that $u = \alpha w$ and $v = \beta w$. Then $x = (\alpha + i\beta)w$ and

$w = (\alpha + i\beta)^{-1}x$ is a real unit eigenvector of every $A \in \mathcal{F}$. Let Q be a real orthogonal matrix whose first column is w and observe that for every $A \in \mathcal{F}, Q^T AQ = \begin{bmatrix} \lambda(A) & * \\ 0 & * \end{bmatrix}$, where $\lambda(A)$ is a real eigenvalue of A. The second possibility is (ii) $\{u, v\}$ is linearly independent. In this event Lemma 2.6 shows how to construct a real nonsingular matrix S such that for every $A \in \mathcal{F}, S^{-1}AS = \begin{bmatrix} A_1(A) & * \\ 0 & * \end{bmatrix}$, where $A_1(A)$ has the form (7). If $b_1(A) \neq 0$, then $a_1(A) \pm ib_1(A)$ is a conjugate pair of non-real eigenvalues of A. If $b_1(A) = 0$, however, then $a_1(A)$ is a double real eigenvalue of A. If $b_1(A) = 0$ for every $A \in \mathcal{F}$ (for example, if every matrix in $\mathcal{F}$ has only real eigenvalues), then split the 2-by-2 block into two 1-by-1 blocks.
(b) Let S be a nonsingular real matrix that has the properties in (a), and let $S = QR$ be a QR factorization (Theorem 2.23 in Lecture 3). In the same way as in the proof of Theorem 2.7, one shows that Q has the asserted properties. □

Remark. Just as in Theorem 2.4, we cannot control the order of appearance of the eigenvalues corresponding to the diagonal blocks in the preceding theorem; we have to take the eigenvalues as they come, according to the common eigenvectors guaranteed by Lemma 1.20 in Lecture 2.

We have shown that any real matrix can be reduced to a real upper quasitriangular form by a real orthogonal similarity. Now we seek to extend this result to a larger set $\mathcal{S} = \{A \in M_n : A\bar{A} = \bar{A}A\}$. Equivalently, this is a set of matrices such that $A\bar{A}$ is real.

Corollary 2.9. *Let $A \in M_n$ and suppose that $A\bar{A} = \bar{A}A$. There is a real orthogonal $Q \in M_n(\mathbf{R})$ and a quasidiagonal $D = J_{n_1} \oplus \cdots \oplus J_{n_m} \in M_n$ such that $Q^T AQ \in M_n$ is a* complex *upper quasitriangular matrix of the form (6) that is partitioned conformally to D and has the following property: For each $j \in \{1, ..., m\}$ such that $n_j = 2$, at least one of $\operatorname{Re} A_j$ or $\operatorname{Im} A_j$ has a conjugate pair of non-real eigenvalues. If each of $\operatorname{Re} A$ and $\operatorname{Im} A$ has only real eigenvalues, then $Q^T AQ \in M_n$ is upper triangular.*

Proof. Write $A = B + iC$, where B and C are real. The hypothesis $A\bar{A} = \bar{A}A$ indicates that $(B+iC)(B-iC) = (B-iC)(B+iC)$, that is, $BC = CB$. It follows from Theorem 2.8 (b) that there exists a real orthogonal $Q \in M_n(\mathbf{R})$ and a quasidiagonal $D = J_{n_1} \oplus \cdots \oplus J_{n_m} \in M_n$ such that each of $Q^T BQ$ and $Q^T CQ$ is a real upper quasitriangular matrix of the form (6) that is partitioned conformally to D. Moreover, for each $j \in \{1, ..., m\}$ such that $n_j = 2$, at least one of $A_j(B)$ or $A_j(C)$ has a conjugate pair of non-real eigenvalues. It follows that $Q^T AQ = Q^T(B + iC)Q = Q^T BQ + iQ^T CQ$ is a complex upper quasitriangular matrix that is partitioned conformally to D. If each of B and C has only real eigenvalues, then every $n_j = 1$ and each of $Q^T BQ$ and $Q^T CQ$ is upper triangular. $\square$

References

[1] Roger A. Horn, Charles R. Johnson (2012) *Matrix Analysis, Second Edition.* Cambridge University Press.