Homework 9 Solution

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Chapter 5. Ex.1 *Corollary 2.3 in Chapter 2 leads to the following simplified version of the Fourier inversion formula. Suppose f is a continuous function supported on an interval $[-M, M]$, whose Fourier transform $\hat{f}$ is of moderate decrease.*
(a) Fix L with $L/2 > M$, and show that $f(x) = \sum a_n(L)e^{\frac{2\pi nx}{L}}$ where

$$a_n(L) = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)e^{-\frac{2\pi nx}{L}}dx = \frac{1}{L}\hat{f}(\frac{n}{L}).$$

Alternatively, we may write $f(x) = \delta \sum_{n=-\infty}^{\infty} \hat{f}(n\delta)e^{2\pi in\delta x}$ with $\delta = \frac{1}{L}$.
(b) Prove that if F is continuous and of moderate decrease, then

$$\int_{-\infty}^{\infty} F(\xi)d\xi = \lim_{\substack{\delta\to 0\\ \delta>0}} \delta \sum_{n=-\infty}^{\infty} F(\delta n).$$

(c) Conclude that $f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi ix\xi}d\xi$.

Proof. (a) Since f is a continuous function supported on an interval $[-M, M]$, i.e., $f(x) = 0$ for all $x \in (-\infty, -M) \cup (M, +\infty)$, we know that the Fourier coefficient of f is
$a_n(L) = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)e^{-\frac{2\pi inx}{L}}dx = \frac{1}{L}\hat{f}(\frac{n}{L})$, where $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ix\xi}dx$ and $\frac{L}{2} > M$.
Next we claim that the Fourier series $\sum_{n=-\infty}^{\infty} a_n(L)e^{\frac{2\pi inx}{L}}$ converges uniformly to $f(x)$.
Since $\hat{f}$ is of moderate decrease, we have $|a_n(L)| \le |\hat{f}(\frac{n}{L})| \le \frac{A}{1+(\frac{n}{L})^2} \le \frac{L^2A}{n^2}$, where A is a constant.
Thus the Fourier series of f is absolutely convergent and by Corollary 2.3 in Chapter 2, we obtain that $f(x) = \sum_{n=-\infty}^{\infty} a_n(L)e^{\frac{2\pi inx}{L}}$.

(b) First we have $|\int_{-\infty}^{\infty} F(\xi)d\xi - \delta \sum_{n=-\infty}^{\infty} F(\delta n)| \le \int_{|x|>N} |F(x)|dx + |\int_{-N}^{N} F(x)dx - \delta \sum_{|n|\le \frac{N}{\delta}} F(\delta n)| + |\delta \sum_{|n|>\frac{N}{\delta}} F(\delta n)| = I_1 + II_2 + III_3$.
Since F is continuous and of moderate decrease, for any $\epsilon > 0$ we can choose $N > 0$ such that $I_1 = \int_{|x|>N} |F(x)|dx < \frac{\epsilon}{3}$. Meanwhile, there exists a $\delta_1 > 0$ such that $II_3 = \delta_1 \sum_{|\delta_1 n|>N} F(n\delta_1) \le \delta_1 \sum_{|x|>N} \frac{A}{1+x^2} \le \int_{|x|>N} \frac{A}{1+x^2}dx < \frac{\epsilon}{3}$, where A is a constant.
Moreover, F is continuous and thus integrable on $[-N, N]$. We can choose $0 < \delta_2 < \delta_1$ such

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that for all $0 < \delta < \delta_2$, $II_2 = |\int_{-N}^{N} F(x)dx - \delta \sum_{|n|\le \frac{N}{\delta}} F(\delta n)| < \frac{\epsilon}{3}$, since $\delta \sum_{|n|\le \frac{N}{\delta}} F(\delta n)$ is almost a Riemann sum of F and may just miss two end points $N, -N$ but the difference is small.
Therefore, for all $0\delta < \delta_2$ we have $|\int_{-\infty}^{\infty} F(\xi)d\xi - \delta \sum_{n=-\infty}^{\infty} F(\delta n)| < \epsilon$ and the result follows.

(c) By Proposition 1.1 (iv), we know that $\hat{f}$ is continuous when $f \in \mathcal{M}(\mathbb{R})$ and f is continuous. Applying (a) and (b) on $F(\xi) = \hat{f}(\xi)e^{2\pi i\xi x}$, we obtain that

$$\int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i\xi x}d\xi = \lim_{\delta\to 0^+} \delta \sum_{n=-\infty}^{\infty} \hat{f}(n\delta)e^{2\pi i n\delta x} = f(x). \qquad \square$$

Chapter 5. Ex.5 *Suppose f is continuous and of moderate decrease.*
(a) Prove that $\hat{f}$ is continuous and $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.
(b) Show that if $\hat{f}(\xi) = 0$ for all ξ, then f is identically 0.

Proof. (a) Since f is of moderate decrease, there exists an $N > 0$ such that $\int_{|x|\ge N} |f(x)|dx < \frac{\epsilon}{4}$ for any $\epsilon > 0$.
Meanwhile, for any $x \in [-N, N]$, due to $\lim_{h\to 0} e^{-2\pi i h x} = 1$, there exists a $\delta > 0$ such that $|e^{-2\pi i h x} - 1| < \frac{\epsilon}{2A}$ when $|h| < \delta$, where $A = \int_{-N}^{N} |f(x)|dx$ for a fixed N.
Thus when $|h| < \delta$, we have

$$\begin{aligned} |\hat{f}(\xi+h) - \hat{f}(\xi)| &\le \int_{|x|\ge N} 2|f(x)|dx + \int_{-N}^{N} |f(x)e^{-2\pi i\xi x}| \cdot |e^{-2\pi i h x} - 1|dx + \\ &< 2\cdot\frac{\epsilon}{4} + (\int_{-N}^{N} |f(x)|dx) \cdot \frac{\epsilon}{2A} \\ &= \epsilon, \end{aligned} \tag{1}$$

showing that $\hat{f}$ is continuous.
In addition, by Riemann-Lebesgue Lemma, for any fixed $N > 0$ we know that $\int_{-N}^{N} f(x)e^{-2\pi i\xi x}dx < \frac{\epsilon}{2}$ when $|\xi|$ is large.
Therefore, $|\hat{f}(\xi)| \le \int_{|x|\ge N} |f(x)|dx + |\int_{-N}^{N} f(x)e^{-2\pi i\xi x}dx| < \epsilon$, yielding that $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.

(b) Since f is of moderate decrease and $g \in \mathcal{S}(\mathbb{R})$, by Theorem 3.1 in the Appendix, we know that the multiplication formula still holds. Then we can choose $\hat{g}(\xi) = K_\delta(t - \xi)$. In reality, $g(x) = \int_{-\infty}^{\infty} \hat{g}(\xi)e^{2\pi i\xi x}d\xi = \int_{-\infty}^{\infty} K_\delta(t-\xi)e^{2\pi i\xi x}d\xi = \int_{-\infty}^{\infty} K_\delta(u)e^{2\pi i(t-u)x}du = e^{2\pi i x t}\hat{K_\delta}(x) = e^{2\pi i x t} \cdot e^{-\pi\delta x^2}$.
Therefore, $0 = \int_{-\infty}^{\infty} f(x)K_\delta(y - x)dx \to f(y)$ uniformly in x as $\delta \to 0$, yielding that f is identically 0. $\square$

Remark: One may want to take $\hat{g}(x) = f(x)$ to prove the results through the continuity of f. However, the case becomes more subtle because f is just of moderate decrease. We need to refer to Fubini's Theorem to write down a rigor proof.

Chapter 5. Ex.7 *Prove that the convolution of two functions of moderate decrease is a function of moderate decrease.*

Proof. First we know that $f * g$ is also continuous as f and g are continuous.
Moreover, by the moderate decreasing property of f and g, we have

$$
\begin{aligned}
|(f*g)(x)| &\leq \int_{|y|\leq \frac{|x|}{2}} |f(x-y)g(y)|dy + \int_{|y|\geq \frac{|x|}{2}} |f(x-y)g(y)|dy \\
&\leq \int_{|y|\leq \frac{|x|}{2}} [\frac{A}{1+(x-y)^2}]|g(y)|dy + \int_{|y|\geq \frac{|x|}{2}} |f(x-y)|(\frac{B}{1+y^2})dy \\
&\leq \frac{4A}{4+x^2}\int_{\infty}^{\infty} |g(y)|dy + \frac{4B}{4+x^2}\int_{\infty}^{\infty} |f(x-y)|dy \\
&\leq \frac{C}{1+x^2},
\end{aligned}
\tag{2}
$$

where A, B, C are constants. $\square$