Homework 8 Solution

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Chapter 4. Ex.4 *Observe that with the definition of ℓ and $\mathcal{A}$ given in the text, the isoperimetric inequality continues to hold (with the same proof) even when Γ is not simple.*
Show that this stronger version of the isoperimetric inequality is equivalent to Wirtinger's inequality, which says that if f is 2π-periodic, of class C^1, and satisfies $\int_0^{2\pi} f(t)dt = 0$, then

$$\int_0^{2\pi} |f(t)|^2 dt \leq \int_0^{2\pi} |f'(t)|^2 dt$$

with equality if and only if $f(t) = A\sin t + B\cos t$.

Proof. ($\Rightarrow$) If the isoperimetric inequality holds, then for any closed curve in $\mathbb{R}^2$ whose length is 2π, we have $\mathcal{A} \leq \pi$, where $\mathcal{A}$ is the area of the region enclosed by this curve.
Choose an appropriate curve γ whose arc-length parametrization satisfies $x'(s) = -y(s)$, $x'(s)^2 + y'(s)^2 = 1$. Then by the periodicity of $x(s)$, we obtain that $\int_0^{2\pi} y(s)ds = -\int_0^{2\pi} x'(s)ds = 0$. Thus,

$$\begin{aligned}
\int_0^{2\pi} (y'(s)^2 - y(s)^2)ds &= \int_0^{2\pi} (x'(s)^2 + y'(s)^2)ds - \int_0^{2\pi} (x'(s)^2 + y(s)^2)ds \\
&= 2\pi - \int_0^{2\pi} (x'(s) + y(s))^2 ds + 2\int_0^{2\pi} x'(s)y(s)ds \\
&= 2\pi - 2\mathcal{A} \\
&\geq 0,
\end{aligned} \tag{1}$$

where we use the fact that $x'(s) = -y(s)$ and $\mathcal{A} = -2\int_0^{2\pi} x'(s)y(s)ds$. Therefore, $\int_0^{2\pi} |y(s)|^2 ds \leq \int_0^{2\pi} |y'(s)|^2 ds$, since $y(s)$ is a real-valued function.
($\Leftarrow$) Conversely, if $\int_0^{2\pi} |f(t)|^2 dt \leq \int_0^{2\pi} |f'(t)|^2 dt$ and f is real-valued, then we can construct g such that $g'(t) = f(t)$ and $g'(t)^2 + f'(t)^2 = 1$.
Then the curve defined by $\gamma(t) = (g(t), f(t))$ is a closed curve in $\mathbb{R}^2$. Thus,

$$\begin{aligned}
2\pi - 2\mathcal{A} &= \int_0^{2\pi} (f'(t)^2 + g'(t)^2)dt + 2\int_0^{2\pi} g'(t)f(t)dt \\
&= \int_0^{2\pi} (g'(t) + f(t))^2 dt + \int_0^{2\pi} (f'(t)^2 - f(t)^2)dt \\
&= \int_0^{2\pi} (f'(t)^2 - f(t)^2)dt \\
&\geq 0,
\end{aligned} \tag{2}$$

yielding the isoperimetric inequality. ☐

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Chapter 4. Ex.7 *Prove the second part of Weyl's criterion: if a sequence of numbers $\xi_1, \xi_2, \ldots$ in $[0,1)$ is equidistributed, then for all $k \in \mathbb{Z} - \{0\}$*

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k \xi_n} \to 0 \ \ \text{as } N \to \infty.$$

Proof. Since the sequence of numbers $\xi_1, \xi_2, \ldots$ in $[0,1)$ is equidistributed, we have

$$\frac{1}{N}\sum_{n=1}^{N} \chi_{[a,b]}(\xi_n) \to \int_a^b \chi_{[a,b]}(x)dx$$

as $N \to \infty$ for any $[a,b] \subset [0,1)$, where $\chi_{[a,b]}(x)$ is the characteristic function of $[a,b]$.
For any continuous function f on $[0,1]$, it should be uniformly continuous on $[0,1]$, i.e., $\forall \epsilon > 0, \exists \delta > 0$, s.t. $\sup |f(x_1) - f(x_2)| < \epsilon$ when $|x_1 - x_2| < \delta$.
Consider a partition of $[0,1]$, say $0 < \frac{1}{N} < \frac{2}{N} < \cdots < \frac{N}{N} = 1$, where $\frac{1}{N} < \delta$.
Define $f_U(x) = \sup_{\frac{j-1}{N} \le y < \frac{1}{N}} f(y)$, $f_L(x) = \inf_{\frac{j-1}{N} \le y < \frac{1}{N}} f(y)$ if $y \in [\frac{j-1}{N}, \frac{j}{N}]$.
Then $f_L(x) \le f(x) \le f_U(x)$ for all $x \in [0,1)$.
We can choose N large enough such that $\int_0^1 f_U(x)dx - \int_0^1 f_L(x)dx < \epsilon$.
Since f_U and f_L are linear combinations of characteristic functions, we obtain that $\frac{1}{N}\sum_{n=1}^{N} f_U(\xi_n) \to \int_0^1 f_U(x)dx$ and $\frac{1}{N}\sum_{n=1}^{N} f_L(\xi_n) \to \int_0^1 f_L(x)dx$ as $N \to \infty$.
Therefore, $\frac{1}{N}\sum_{n=1}^{N} f(\xi_n) \to \int_0^1 f(x)dx$ as $N \to \infty$ for any continuous function f.
Particularly, $\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k \xi_n} \to 0 = \int_0^1 e^{2\pi i k x}dx$, for all $k \in \mathbb{Z} - \{0\}$ □

Chapter 4. Ex.10 *Suppose that f is a periodic function on $\mathbb{R}$ of periodic 1, and $\{\xi_n\}$ is a sequence which is equidistributed in $[0,1)$. Prove that:*
(a) If f is continuous and satisfies $\int_0^1 f(x)dx = 0$, then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x+\xi_n) = 0 \ \ \text{uniformly in } x.$$

(b) If f is merely integrable on $[0,1]$ and satisfies $\int_0^1 f(x)dx = 0$, then

$$\lim_{N\to\infty} \int_0^1 |\frac{1}{N}\sum_{n=1}^{N} f(x+\xi_n)|^2 dx = 0$$

Proof. (a) Due to the equidistributed property of $\{\xi_n\}$, we know that $\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k \xi_n} \to 0$ for all $k \in \mathbb{Z} - \{0\}$ by the Exercise 7.

Since $|e^{2\pi ikx}|=1$ and $\int_0^1 e^{2\pi ikx}dx=0$ for all $k\in\mathbb{Z}-\{0\}$, we have $\frac{1}{N}\sum\limits_{n=1}^{N} e^{2\pi ik(\xi_n+x)}\to 0$
Thus, the result holds for any trigonometric polynomials, i.e., for any $\epsilon>0$, there exists an $N(\epsilon)>0$ such that $|\frac{1}{N}\sum\limits_{n=1}^{N} P(x+\xi_n)|<\frac{\epsilon}{2}$, where $P(x)$ is any polynomial whose constant term is 0.
By Corollary 5.4 in Chapter 2, we can choose a trigonometric polynomial $P(x)$, whose constant term is 0, such that $\sup\limits_{x\in\mathbb{R}}|P(x)-f(x)|<\frac{\epsilon}{2}$ for any $\epsilon>0$.
Thus, $|\frac{1}{N}\sum\limits_{n=1}^{N} f(x+\xi_n)|\leq\frac{1}{N}\sum\limits_{n=1}^{N}|f(x+\xi_n)-P(x+\xi_n)|+|\frac{1}{N}\sum\limits_{n=1}^{N}P(x+\xi_n)|<\epsilon$ when $N(\epsilon)$ is large.
Therefore, $\lim\limits_{N\to\infty}|\frac{1}{N}\sum\limits_{n=1}^{N} f(x+\xi_n)|=0$ uniformly in x.

(b) By Lemma 1.5 in the Appendix, we know that there exists a sequence $\{g_k\}_{k=1}^{\infty}$ of continuous functions such that $\sup\limits_{x\in[0,1]}|g_k(x)|\leq B$ and $\int_0^1|f(x)-g_k(x)|dx\to 0$ as $k\to\infty$, where $B=\sup\limits_{x\in[0,1]}|f(x)|$. Thus,

$$
\begin{aligned}
\int_0^1|\frac{1}{N}\sum_{n=1}^{N} f(x+\xi_n)|^2dx &\leq \int_0^1|\frac{1}{N}\sum_{n=1}^{N}(f(x+\xi_n)-g_k(x+\xi_n))|^2dx+\int_0^1|\frac{1}{N}\sum_{n=1}^{N} g_k(x+\xi_n)|^2dx\\
&\leq \frac{1}{N}\sum_{n=1}^{N}\int_0^1|(f(x+\xi_n)-g_k(x+\xi_n))|^2dx+\int_0^1|\frac{1}{N}\sum_{n=1}^{N} g_k(x+\xi_n)|^2dx\\
&\to 0,\ \text{as } k\to\infty, N\to\infty,
\end{aligned}
\tag{3}
$$

where $|\frac{1}{N}\sum\limits_{n=1}^{N} g_k(x+\xi_n)|^2=0$, $k=1,2,...$ by (a). □