Homework 7 Solution

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Chapter 3. Ex.11 *The inequalities of Wirtinger and Poincaré establish a relationship between the norm of a function and that of its derivative.*
(a) If f is T-periodic, continuous, and piecewise C^1 with $\int_0^T f(t)dt = 0$, show that

$$\int_0^T |f(t)|^2 dt \leq \frac{T^2}{4\pi^2} \int_0^T |f'(t)|^2 dt,$$

with equality if and only if $f(t) = A\, sin(2\pi t/T) + B\, cos(2\pi t/T)$.
(b) If f is as above and g is just C^1 and T-periodic, prove that

$$|\int_0^T \overline{f(t)} g(t) dt|^2 \leq \frac{T^2}{4\pi^2} \int_0^T |f(t)|^2 dt \int_0^T |g'(t)| dt.$$

(c) For any compact interval $[a, b]$ and any continuously differentiable function f with $f(a) = f(b) = 0$, show that

$$\int_a^b |f(t)|^2 dt \leq \frac{(b-a)^2}{\pi^2} \int_a^b |f'(t)|^2 dt.$$

Discuss the case of equality, and prove that the constant $\frac{(b-a)^2}{\pi^2}$ cannot be improved.

Proof. (a) The condition $\int_0^T f(t)dt = 0$ implies that $\hat{f}(0) = 0$.
By the smoothness and the periodicity of f, we know that

$$\begin{aligned} \hat{f}(n) &= \frac{1}{T} \int_0^T f(t) e^{-\frac{2\pi nit}{T}} dt \\ &= \frac{1}{T}[f(0^+) - f(T^-)] \frac{T}{2\pi in} + \frac{T}{2\pi in} \cdot \frac{1}{T} \int_0^T f'(t) e^{-\frac{2\pi int}{T}} dt \\ &= \frac{T}{2\pi in} \hat{f'}(n). \end{aligned} \tag{1}$$

Thus, by Parseval's identity,

$$\begin{aligned} \int_0^T |f(t)|^2 dt = T \sum_{|n|>0} |\hat{f}(n)|^2 &= \frac{T^3}{4\pi^2} \sum_{|n|>0} \frac{|\hat{f'}(n)|^2}{n^2} \\ &\leq \frac{T^3}{4\pi^2} \sum_{|n|>0} |\hat{f'}(n)|^2 = \frac{T^3}{4\pi^2} \cdot \frac{1}{T} \int_0^T |f'(t)|^2 dt = \frac{T^2}{4\pi^2} \int_0^T |f'(t)|^2 dt. \end{aligned} \tag{2}$$

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The equality holds if and only if $\hat{f}(n) = 0$ for all $n > 1$, i.e.,
$f(t) = a_1 e^{\frac{2\pi it}{T}} + a_{-1} e^{\frac{2\pi it}{T}} = Asin(\frac{2\pi t}{T}) + Bcos(\frac{2\pi t}{T})$.

(b) With the same conditions on f, we still have $\hat{f}(0) = 0$.
Meanwhile, the equality $\hat{g}(n) = \frac{T}{2\pi in}\hat{g}'(n)$ holds when $n \neq 0$.
By Lemma 1.5 and Paserval's equality, we obtain that

$$\begin{aligned}
|\int_0^T \overline{f(t)}g(t)dt|^2 = T^2|\sum_{|n|\geq 0}\overline{\hat{f}(n)}\hat{g}(n)|^2 &= T^2|\sum_{|n|>0}\overline{\hat{f}(n)}\hat{g}(n)|^2 \\
&\leq (T\sum_{|n|>0}|\hat{f}(n)|^2)(T\sum_{|n|>0}|\hat{g}(n)|^2) \\
&= \int_0^T |f(t)|^2 dt \cdot \frac{T^3}{4\pi^2}\sum_{|n|>0}\frac{|\hat{g}'(n)|^2}{n^2} \qquad (3) \\
&\leq \int_0^T |f(t)|^2 dt \cdot \frac{T^3}{4\pi^2}\sum_{|n|>0}|\hat{g}'(n)|^2 \\
&= \frac{T^2}{4\pi^2}\int_0^T |f(t)|^2 dt \int_0^T |g'(t)|dt.
\end{aligned}$$

Remark: Please figure out the correctness of every equalities and inequalities by yourself.

(c) To extend f to be odd with respect to a, we define F as follows,

$$F(t) = \begin{cases} f(t) & \text{if } t \in [a, b], \\ -f(2a - t) & \text{if } t \in [2a - b, a). \end{cases}$$

Then we can extend F on the real line such that it is $2(b - a)$-periodic.
Then by definition of F, we know that F is piecewise C^1, $\int_0^{2(b-a)} F(t)dt = 0$, and $F'(a + h) = F'(a - h)$, where $h \in \mathbb{R}$.
By (a) and the symmetry of F, we obtain that

$$\int_a^b |f(t)|^2 dt = \frac{1}{2}\int_0^{2(b-a)} |F(t)|^2 dt \leq \frac{(b-a)^2}{2\pi^2}\int_0^{2(b-a)} |F'(t)|^2 dt = \frac{(b-a)^2}{\pi^2}\int_a^b |f'(t)|^2 dt.$$

Since $F(t - a)$ is an odd function, the equality holds if and only if $f(t) = F(t) = Asin(\frac{\pi t}{b-a})$. $\square$

Chapter 3. Ex.16 *Let f be a 2π-periodic function which satisfies a Lipschitz condition with constant K; that is,*

$$|f(x) - f(y)| \leq K|x - y| \text{ for all } x, y.$$

This is simply the Hölder condition with $\alpha = 1$, so by the previous exercise, we see that $\hat{f}(n) = O(1/|n|)$. Since the harmonic series $\sum 1/n$ diverges, we cannot say anything (yet)

about the absolute convergence of the Fourier series of f. The outline below actually proves that the Fourier series of f converges absolutely and uniformly.
(a) For every positive h we define $g_h(x) = f(x+h) - f(x-h)$. Prove that

$$\frac{1}{2\pi}\int_0^{2\pi} |g_h(x)|^2 dx = \sum_{n=-\infty}^{\infty} 4|sin\, nh|^2 |\hat{f}(n)|^2,$$

and show that

$$\sum_{n=-\infty}^{\infty} |sin\, nh|^2 |\hat{f}(n)|^2 \leq K^2 h^2.$$

(b) Let p be a positive integer. By choosing $h = \frac{\pi}{2^{p+1}}$, show that

$$\sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)|^2 \leq \frac{K^2\pi^2}{2^{2p+1}}.$$

(c) Estimate $\sum\limits_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)|$, and conclude that the Fourier series of f converges absolutely, hence uniformly.
(d) In fact, modify the argument slightly to prove Bernstein's theorem: If f satisfies a Hölder condition of order $\alpha > \frac{1}{2}$, then the Fourier series of f converges absolutely.

Proof. (a) First we have

$$\begin{aligned} \hat{g}_h(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+h)e^{-inx}dx - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-h)e^{-inx}dx \\ &= \frac{1}{2\pi}\int_{-\pi+h}^{\pi+h} f(u)e^{-inu}\cdot e^{inh}du - \frac{1}{2\pi}\int_{-\pi-h}^{\pi-h} f(u)e^{-inu}\cdot e^{-inh}du \\ &= e^{inh}\hat{f}(n) - e^{-inh}\hat{f}(n) \\ &= [2isin(nh)]\hat{f}(n). \end{aligned} \tag{4}$$

Applying Parseval's identity, we obtain that $\frac{1}{2\pi}\int_0^{2\pi} |g_h(x)|^2 dx = \sum\limits_{n=-\infty}^{\infty} 4|sin(nh)|^2 |\hat{f}(n)|^2$.

Since $|g_h(x)| = |f(x+h) - f(x-h)| \leq 2Kh$, the inequality $\sum\limits_{n=-\infty}^{\infty} |sin\, nh|^2 |\hat{f}(n)|^2 \leq K^2 h^2$ follows.

(b) The conditions, $h = \frac{\pi}{2^{p+1}}$ and $2^{p-1} < |n| \leq 2^p$, imply that $\frac{\pi}{4} < |n|h \leq \frac{\pi}{2}$ and thus $|sin\, nh|^2 \geq \frac{1}{2}$.

Therefore, $\sum\limits_{2^{p-1}<|n|\leq 2^p} 1\cdot |\hat{f}(n)|^2 \leq 2\sum\limits_{n=-\infty}^{\infty} |sin\, nh|^2 |\hat{f}(n)|^2 \leq \frac{K^2\pi^2}{2^{2p+1}}$, where we use the fact that $|sin\, nh|^2 |\hat{f}(n)|^2 \geq 0$ and $h = \frac{\pi}{2^{p+1}}$.

(c) By Cauchy-Schwartz inequality, we have

$$\begin{aligned}\sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)| &\leq \Big(\sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)|^2\Big)^{\frac{1}{2}} \cdot \Big(\sum_{2^{p-1}<|n|\leq 2^p} 1^2\Big)^{\frac{1}{2}} \\ &\leq \frac{K\pi}{2^{p+\frac{1}{2}}} \cdot 2^{\frac{p-1}{2}} \\ &= \frac{K\pi}{2^{\frac{p}{2}+1}}.\end{aligned} \tag{5}$$

Thus, $\sum\limits_{0\leq|n|<\infty} |\hat{f}(n)| = \hat{f}(0) + \sum\limits_{p=1}^{\infty}[\sum\limits_{2^{p-1}<|n|\leq 2^p} \hat{f}(n)] \leq \hat{f}(0) + \sum\limits_{p=1}^{\infty} \frac{K\pi}{2(\sqrt{2})^p} < \infty$, showing that the Fourier series of f converges absolutely, hence uniformly.

(d) Again, we define $g_h(x) = f(x+h) - f(x-h)$, where h is positive.

Since $|g_h(x)| \leq K \cdot 2^{\alpha} h^{\alpha}$, we know that $\sum\limits_{n=-\infty}^{\infty} |sin\, nh|^2 |\hat{f}(n)|^2 \leq K^2 \cdot 2^{2\alpha-2} h^{2\alpha}$.

We still choose h to be $\frac{\pi}{2^{p+1}}$ and obtain that $\sum\limits_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)|^2 \leq \frac{K^2\pi^{2\alpha}}{2^{2\alpha p+1}}$ by the same arguments.

Then, by Cauchy-Schwartz inequality, we have $\sum\limits_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)| \leq (\frac{K^2\pi^{2\alpha}}{2^{2\alpha p+1}})^{\frac{1}{2}} (2^{p-1})^{\frac{1}{2}} = \frac{K\pi^{\alpha}}{2(2^{\alpha-\frac{1}{2}})^p}$.

Therefore, if $\alpha > \frac{1}{2}$, then $\sum\limits_{0\leq|n|<\infty} |\hat{f}(n)| \leq \hat{f}(0) + \sum\limits_{p=1}^{\infty} \frac{K\pi^{\alpha}}{2(2^{\alpha-\frac{1}{2}})^p} < \infty$. $\square$