Homework 5 Solution

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Chapter 2. Ex.13 *The purpose of this exercise is to prove that Abel summability is stronger than the standard or Cesáro methods of summation.*
(a) Show that if the series $\sum_{n=1}^{\infty} c_n$ *of complex numbers converges to a finite limit* s*, then the series is Abel summable to* s*.*
(b) However, show that there exist series which are Abel summable, but that do not converge.
(c) Argue similarly to prove that if a series $\sum_{n=1}^{\infty} c_n$ *is Cesáro summable to* σ*, then it is Abel summable to* σ*.*
(d) Give an example of a series that is Abel summable but not Cesáro summable.
The results above can be summarized by the following implications about series:

$$\textit{convergent} \implies \textit{Ceráro summable} \implies \textit{Abel summable},$$

and the fact that none of the arrows can be reversed.

Proof. (a) Without loss of generality, we assume that $s = 0$. In fact, if $\sum_{n=1}^{\infty} c_n = s \neq 0$, we can simply define $d_n = c_n - \frac{s}{2^n}$ and $\sum_{n=1}^{\infty} d_n = 0$.
Letting $s_1 = c_1$, $s_N = c_1 + c_2 + \cdots + c_N$, we have

$$\begin{aligned} \sum_{n=1}^{N} c_n r^n &= \sum_{n=1}^{N-1} (s_{n+1} - s_n) r^{n+1} + c_1 r \\ &= \sum_{n=1}^{N} s_n r^n - \sum_{n=1}^{N-1} s_n r^{n+1} \\ &= (1-r) \sum_{n=1}^{N} s_n r^n + s_N r^{N+1}. \end{aligned} \tag{1}$$

With the assumption that $s = 0$, we will obtain that $\sum_{n=1}^{\infty} c_n r^n = (1-r) \sum_{n=1}^{\infty} s_n r^n$ as $N \to \infty$ in the preceding equation.
For any $\epsilon > 0$, we may choose N large enough such that $|s_n| < \epsilon$ when $n > N$. Also, there exists an $M > 0$ such that $\sup_{n\in\mathbb{N}} |s_n| \leq M$.

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Then we have

$$
\begin{aligned}
|\sum_{n=1}^{\infty} c_n r^n| &= |(1-r)\sum_{n=1}^{\infty} s_n r^n| \\
&\le |(1-r)\sum_{n=1}^{N} s_n r^n| + |(1-r)\sum_{n=N+1}^{\infty} s_n r^n| \\
&< (1-r)\sum_{n=1}^{N} M r^n + (1-r)\frac{\epsilon r^N}{1-r} \\
&\le Mr(1-r^N) + \epsilon.
\end{aligned} \tag{2}
$$

Thus, $\lim_{r\to 1^-}\sup|\sum_{n=1}^{\infty} c_n r^n| \le \epsilon$. Since ϵ is arbitrary, we have $\lim_{r\to 1^-}\sum_{n=1}^{\infty} c_n r^n = 0$.

(b) Consider $c_n = (-1)^n$. The partial sum $s_N = \sum_{n=1}^{N} c_n$ is -1 when N is odd and 0 when N is even. Thus it does not converge. However, its Abel limit is $\lim_{r\to 1^-}\sum_{n=1}^{\infty}(-r)^n = \lim_{r\to 1^-}\frac{-r}{1+r} = -\frac{1}{2}$.

(c) First we let $\sigma_N = \frac{s_1+s_2+\cdots+s_N}{N}$ and obtain that $s_1 = \sigma_1$, $s_N = N\sigma_N - (N-1)\sigma_{N-1}$.
Then assuming $\sigma = 0$ and by (a), we obtain that

$$
\begin{aligned}
\sum_{n=1}^{N} c_n r^n &= (1-r)\sum_{n=1}^{N} s_n r^n + s_N r^{N+1} \\
&= (1-r)\sum_{n=1}^{N}[n\sigma_n - (n-1)\sigma_{n-1}]r^n + [N\sigma_N - (N-1)\sigma_{N-1}]r^{N+1} \\
&= (1-r)\sum_{n=1}^{N} n\sigma_n r^n - (1-r)\sum_{n=1}^{N-1} n\sigma_n r^n + [N\sigma_N - (N-1)\sigma_{N-1}]r^{N+1} \\
&(1-r)^2\sum_{n=1}^{N} n\sigma_n r^n + (2-r)N\sigma_N r^{N+1} - (N-1)\sigma_{N-1}r^{N+1}.
\end{aligned} \tag{3}
$$

Since $\{\sigma_n\}$ is bounded and the series $\sum_{n=1}^{\infty} r^n$ converges for any $0 < r < 1$, the derivative of $\sum_{n=1}^{\infty} r^n$ with respect to r also converges, yielding that the single items Nr^{N-1} and $(N-1)r^{N-2}$ tend to 0 as $N \to \infty$.
Letting $N \to \infty$ in the previous equation, we have $\sum_{n=1}^{\infty} c_n r^n = (1-r)^2\sum_{n=1}^{\infty} n\sigma_n r^n$.
With the assumption $\sigma = 0$, we can choose N large enough such that $|\sigma_n| < \epsilon$ when $n > N$ for

any $\epsilon > 0$. Meanwhile, there exists a $B > 0$ such that $|\sigma_n| \leq B$ for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}
|\sum_{n=1}^{\infty} c_n r^n| &= |(1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n| \\
&\leq (1-r)^2 \sum_{n=1}^{N} Bnr^n + (1-r)^2 \sum_{n=N+1}^{\infty} \epsilon n r^n \\
&= (1-r)^2 B \frac{r-(N+1)r^{N+1}+Nr^{N+2}}{(1-r)^2} + (1-r)^2 \epsilon \frac{(N+2)r^{N+1}-(1+N)r^{N+2}}{(1-r)^2} \\
&= B[r-(N+1)r^{N+1}+Nr^{N+2}] + \epsilon[(N+2)r^{N+1}-(1+N)r^{N+2}].
\end{aligned} \tag{4}$$

As $r \to 1^-$, we have $\lim_{r\to 1^-} \sup |\sum_{n=1}^{\infty} c_n r^n| \leq \epsilon$, i.e., $\lim_{r\to 1^-} \sum_{n=1}^{\infty} c_n r^n = 0$ by the arbitrariness of ϵ.
In the case when $\sigma \neq 0$, we let $d_1 = c_1 - \sigma$, $d_n = c_n$ for $n > 1$.
Then $\lim_{r\to 1^-} \sum_{n=1}^{\infty} d_n r^n = 0 = \lim_{r\to 1^-} \sum_{n=1}^{\infty} c_n r^n - \sigma$.

(d) Consider $c_n = (-1)^{n-1} n$. Then $\sum_{n=1}^{\infty} (-1)^{n-1} n r^n = -[\sum_{n=1}^{\infty} (n+1)(-r)^n - \sum_{n=1}^{\infty} (-r)^n] = \frac{r}{(1+r)^2}$.
Thus its Abel limit is $\lim_{r\to 1^-} \sum_{n=1}^{\infty} (-1)^{n-1} r^n = \lim_{r\to 1^-} \frac{r}{(1+r)^2} = \frac{1}{4}$.
Note that the Cesáro sum has the property $\sigma_n - (\frac{n-1}{n})\sigma_{n-1} = \frac{a_n}{n}$. Hence for a Cesáro summable series $\sum_{n=1}^{\infty} a_n$, $\lim_{n\to\infty} \frac{a_n}{n}$ must be 0.
Therefore, $\sum_{n=1}^{\infty} (-1)^{n-1} n$ is not Cesáro summable. $\square$