Homework 3 Solution

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Chapter 2. Problem 1 *One can construct Riemann integrable functions on* $[0,1]$ *that have a dense set of discontinuities as follows.*
(a) Let $f(x) = 0$ *when* $x < 0$, *and* $f(x) = 1$ *if* $x \geq 0$. *Choose a countable dense sequence* $\{r_n\}$ *in* $[0,1]$. *Then, show that the function*

$$F(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} f(x - r_n)$$

is integrable and has discontinuities at all points of the sequence $\{r_n\}$.
(b) Consider next

$$F(x) = \sum_{n=1}^{\infty} 3^{-n} g(x - r_n),$$

where $g(x) = sin\, 1/x$ *when* $x \neq 0$, *and* $g(0) = 0$. *Then* F *is integrable, discontinuous at each* $x = r_n$, *and fails to be monotonic in any subinterval of* $[0,1]$.
(c) The original example of Riemann is the function

$$F(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2},$$

where $(x) = x$ *for* $x \in (-1/2, 1/2]$ *and* (x) *is continued to* $\mathbb{R}$ *by periodicity, that is,* $(x+1) = (x)$. *It can be shown that* F *is discontinuous whenever* $x = m/2n$, *where* $m, n \in \mathbb{Z}$ *with* m *odd and* $n \neq 0$.

Proof. (a) First we have $|F(x)| \leq \sum\limits_{n=1}^{\infty} \frac{1}{n^2} < \infty$, yielding that F is bounded.
Then $F(x)$ is nondecreasing because $f(x)$ is nondecreasing.
Thus, by the Proposition 1.3 in the Appendix, we know that $F(x)$ is integrable.
For any countable dense set $\{r_n\}$ in $[0,1]$, like $\mathbb{Q} \cap [0,1]$, it suffices to prove that F is discontinuous at an arbitrary point r_k selected from the reordered set $\{r_n\}$, where $r_k < r_{k+1}$, $k = 1, 2, ...$
The left limit of F at r_k is $\lim\limits_{x \to r_k^-} F(x) = \lim\limits_{x \to r_k^-} \sum\limits_{n=1}^{\infty} \frac{1}{n^2} f(x - r_n) = \sum\limits_{n=1}^{k-1} \frac{1}{n^2}$, while the right limit of F at the same point is $\lim\limits_{x \to r_k^+} F(x) = \lim\limits_{x \to r_k^+} \sum\limits_{n=1}^{\infty} \frac{1}{n^2} f(x - r_n) = \sum\limits_{n=1}^{k} \frac{1}{n^2} \neq \lim\limits_{x \to r_k^-} F(x)$.
Therefore, F has discontinuities as all points of the sequence $\{r_n\}$.

(b) Similarly, we have $|F(x)| \leq \sum\limits_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2}$, showing that F is bounded.
Moreover, since $g(x) = sin\frac{1}{x}$ is discontinuous only at the point $x = 0$, for each $n \in \mathbb{Z}^+$,

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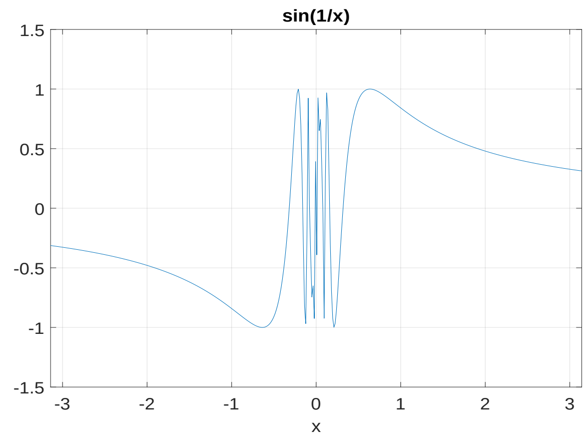
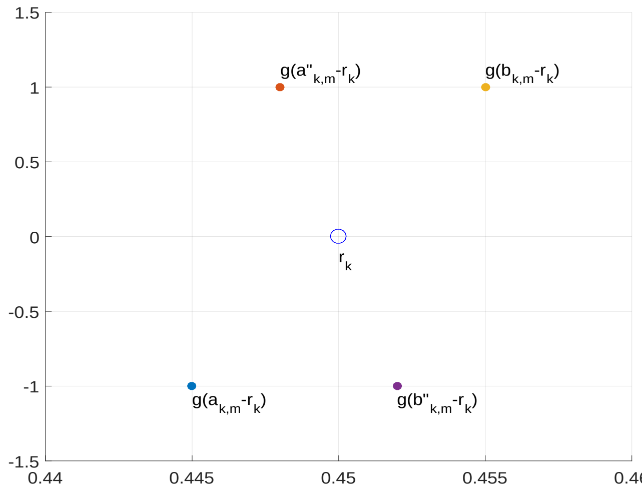
$3^{-n}g(x-r_n)$ has a unique discontinuity at r_n. Together with Lemma 1.6 in the Appendix and the fact that $\{r_n\}$ is countable, we conclude that the discontinuities of F has measure 0 and thus F is integrable.
Next we want to prove that F is discontinuous at each $x = r_n$ and fails to be monotonic in any subinterval of $[0,1]$, by leveraging the fluctuation of $sin\frac{1}{x}$ around 0. See Figure (a) for a visualization.



(a) Plotting for $sin\,\frac{1}{x}$



(b) An Example of the Configuration of these Four Points

Since $\{r_n\}$ is dense in $[0,1]$, any subinterval of $[0,1]$ contains at least one element in $\{r_n\}$, says r_k.
Consider four numerical sequences $a_{k,m} = r_k - \frac{1}{\frac{\pi}{2}+2m\pi}$, $a'_{k,m} = r_k - \frac{1}{\frac{3\pi}{2}+2m\pi}$, $b_{k,m} = r_k + \frac{1}{\frac{\pi}{2}+2m\pi}$, and $b'_{k,m} = r_k + \frac{1}{\frac{3\pi}{2}+2m\pi}$. See Figure (b) for a visualization.
Then $g(a_{k,m}-r_k) = g(b'_{k,m}-r_k) = -1$, and $g(a'_{k,m}-r_k) = g(b_{k,m}-r_k) = 1$.
We respectively divide the summation in $F(a_{k,m})$ and $F(b_{k,m})$ into three parts as follows,

$$F(a_{k,m}) = \sum_{n=1}^{k-1} 3^{-n}g(a_{k,m}-r_n) \ + \ 3^{-k}g(a_{k,m}-r_k) \ + \sum_{n=k+1}^{\infty} 3^{-n}g(a_{k,m}-r_n) = I_1^a \ + \ II_2^a \ + \ III_3^a$$

$$F(b_{k,m}) = \sum_{n=1}^{k-1} 3^{-n}g(b_{k,m}-r_n) \ + \ 3^{-k}g(b_{k,m}-r_k) \ + \sum_{n=k+1}^{\infty} 3^{-n}g(b_{k,m}-r_n) = I_1^b \ + \ II_2^b \ + \ III_3^b.$$

Since $\lim\limits_{m\to\infty} a_{k,m} = \lim\limits_{m\to\infty} b_{k,m} = r_k$, we obtain that $I_1^a = I_1^b$ when m is sufficiently large.
Using the fact that $3^{-k} > \sum\limits_{n>k} 3^{-n}$ and $g(a_{k,m}-r_k) = -1$, we know that

$$II_2^a \ + \ III_3^a \leq -3^{-k} + |\sum_{n=k+1}^{\infty} 3^{-n}g(a_{k,m}-r_n)| < -3^{-k} + \sum_{n>k} 3^{-n} < 0.$$

On the other hand, we also have

$$II_2^b + III_3^b \geq 3^{-k} - |\sum_{n=k+1}^{\infty} 3^{-n} g(a_{k,m} - r_n)| > 3^{-k} - \sum_{n>k} 3^{-n} > 0.$$

Therefore, we obtain that $F(a_{k,m}) < F(b_{k,m})$.
Likewise, we can prove that $F(a_{k,m}) < F(a'_{k,m})$ and $F(a'_{k,m}) > F(b'_{k,m})$ with the same argument. Since $a_{k,m} < a'_{k,m} < r_k < b'_{k,m} < b_{k,m}$ for each m and $|a_{k,m} - b_{k,m}| \to 0$ as $m \to 0$, F is discontinuous at r_k and fails to be monotonic in $[a_{k,m}, b_{k,m}$ and thus in any subinterval of $[0, 1]$.

(c) Let $x_m = \frac{m}{2n}$, where $m, n \in \mathbb{Z}$ with m odd and $n \neq 0$.
Then $|F(x_m + 0) - F(x_m - 0)| = \frac{1}{2m^2} > 0$. Thus F is discontinuous at x_m. □