Homework 2 Solution

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Chapter 2. Ex.2 *In this exercise we show how the symmetries of a function imply certain properties of its Fourier coefficients. Let f be a 2π-periodic Riemann integrable function defined on $\mathbb{R}$.*
(a) Show that the Fourier series of the function f can be written as

$$f(\theta) \sim \hat{f}(0) + \sum_{n\geq 1} [\hat{f}(n) + \hat{f}(-n)] cosn\theta + i[\hat{f}(n) - \hat{f}(-n)] sinn\theta.$$

(b) Prove that if f is even, then $\hat{f}(n) = \hat{f}(-n)$, and we get a cosine series.
(c) Prove that if f is odd, then $\hat{f}(n) = -\hat{f}(-n)$, and we get a sine series.
(d) Suppose that $f(\theta + \pi) = f(\theta)$ for all $\theta \in \mathbb{R}$. Show that $\hat{f}(n) = 0$ for all odd n.
(e) Show that f is real-valued if and only if $\overline{\hat{f}(n)} = \hat{f}(-n)$ for all n.

Proof. (a)

$$\begin{aligned}
f(\theta) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{in\theta} &= \hat{f}(0) + \sum_{n=1}^{\infty} [\hat{f}(n) e^{in\theta} + \hat{f}(-n) e^{-in\theta}] \\
&= \hat{f}(0) + \sum_{n=1}^{\infty} [(\hat{f}(n) + \hat{f}(-n)) cos\, n\theta + i(\hat{f}(n) - \hat{f}(-n)) sin\, n\theta] \qquad (1) \\
&= \hat{f}(0) + \sum_{n=1}^{\infty} (A_n cos\, n\theta + B_n sin\, n\theta),
\end{aligned}$$

where $A_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s)(e^{-ins} + e^{ins}) ds = \frac{1}{\pi} \int_{-\pi}^{\pi} f(s) cos\, nsds$,
$B_n = \frac{i}{2\pi} \int_{-\pi}^{\pi} f(s)(e^{-ins} - e^{ins}) ds = \frac{1}{\pi} \int_{-\pi}^{\pi} f(s) sin\, nsds$.

Notes: This result coincides with the definition in our Mathematical Analysis textbook.

(b) If f is even, then $f(s) sin\, ns$ is odd. Thus $B_n = i[\hat{f}(n) - \hat{f}(-n)] = 0$, i.e., $\hat{f}(n) = \hat{f}(-n)$ and $f(\theta) \sim \hat{f}(0) + 2 \sum_{n=1}^{\infty} \hat{f}(n) cos\, n\theta$.

(c) If f is odd, then $f(s) cos\, ns$ is odd. Thus $A_n = \hat{f}(n) + \hat{f}(-n) = 0$, i.e., $\hat{f}(n) = \hat{f}(-n)$ and $f(\theta) \sim 2i \sum_{n=1}^{\infty} \hat{f}(n) sin\, n\theta$.

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(d) If $f(\theta+\pi)=f(\theta)$ for all $\theta\in\mathbb{R}$, then

$$\begin{aligned}\hat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{0} f(t)e^{-int}dt + \frac{1}{2\pi}\int_{0}^{\pi} f(t)e^{-int}dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{0} f(t)e^{-int}dt + \frac{1}{2\pi}\int_{-\pi}^{0} f(\theta+\pi)e^{-in\theta}\dot{e}^{in\pi}d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{0} f(t)e^{-int}(1+e^{in\pi})dt.\end{aligned} \tag{2}$$

Since $e^{in\pi}=-1$ when n is odd, $\hat{f}(n)=0$ for all odd n.

(e) If f is real-valued, then $\overline{f(\theta)}=f(\theta)$.

Thus, $\overline{\hat{f}(n)}=\overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt}=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{int}dt=\hat{f}(-n)$.

Conversely, we need an additional condition on f, that is, at least the imaginary part of f should be continuous on $\mathbb{R}$. If $\overline{\hat{f}(n)}=\hat{f}(-n)$ for all n, then

$$\overline{f(\theta)} \sim \overline{\sum_{n=-\infty}^{\infty}\hat{f}(n)e^{in\theta}} = \sum_{n=-\infty}^{\infty}\overline{\hat{f}(n)}e^{-in\theta} = \sum_{n=-\infty}^{\infty}\hat{f}(-n)e^{-in\theta} = \sum_{n=-\infty}^{\infty}\hat{f}(n)e^{in\theta} \sim f(\theta).$$

Therefore, $f(\theta)$ and $\overline{f(\theta)}$ have identical Fourier coefficients. By Parseval's identity, we have $\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta)-\overline{f(\theta)}|^2d\theta=\frac{2}{\pi}\int_{-\pi}^{\pi}|v(\theta)|^2d\theta$, where $v(\theta)$ is the imaginary part of $f(\theta)$. With the continuous assumption on v, it becomes natural that $v(\theta)=0$ for all $\theta\in\mathbb{R}$. $\square$

Chapter 2. Ex.9 *Let $f(x)=\chi_{[a,b]}(x)$ be the characteristic function of the interval $[a,b]\subset[-\pi,\pi]$, that is,*

$$\chi_{[a,b]}(x)=\begin{cases}1 & \text{if } x\in[a,b],\\ 0 & \text{otherwise.}\end{cases}$$

(a) Show that the Fourier series of f is given by

$$f(x)\sim\frac{b-a}{2\pi}+\sum_{n\neq 0}\frac{e^{-ina}-e^{-inb}}{2\pi in}e^{inx}.$$

The sum extends over all positive and negative integers excluding 0.

(b) Show that if $a\neq-\pi$ or $b\neq\pi$ and $a\neq b$, then the Fourier series does not converge absolutely for any x.

(c) However, prove that the Fourier series converges at every point x. What happen if $a=-\pi$ and $b=\pi$?

Proof.(a) $\hat{f}(0)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\chi_{[a,b]}(x)dx=\frac{1}{2\pi}\int_a^b dx=\frac{b-a}{2\pi}$,

$\hat{f}(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\chi_{[a,b]}(x)e^{-inx}dx=\frac{1}{2\pi}\int_a^b e^{-inx}dx=\frac{e^{-ina}-e^{-inb}}{2\pi in}$ for $n\neq 0$.

(b) First we have $|e^{-ina} - e^{-inb}| = |e^{-ina}||1 - e^{in(a-b)}| = \sqrt{2 - 2cos\, n(a-b)} = 2|sin\frac{n(b-a)}{2}|$.

Thus, $|\frac{e^{-ina}-e^{-inb}}{2\pi in}e^{inx}| = \frac{|sin\, n\theta_0|}{\pi n} \geq \frac{c}{\pi n}$, where $\theta_0 = \frac{b-a}{2} \in (0, \pi)$.

Since $\sum_{n=1}^{\infty} \frac{c}{\pi n}$ diverges, we can conclude that the Fourier series does not converge absolutely for any x.

(c) $f(x) \sim \frac{b-a}{2\pi} + \sum_{n\neq 0} \frac{e^{-ina}-e^{-inb}}{2\pi in}e^{inx} = \frac{b-a}{2\pi} + \sum\limits_{n=1}^{\infty} \frac{1}{n\pi}[sin\, n(x-a) - sin\, n(x-b)]$.

Let $a_n = \frac{1}{n\pi}$, $b_n = sin\, n(x-a)$, $c_n = sin\, n(x-b)$.

Since $\sum\limits_{k=1}^{n} b_k = \frac{1}{2sin\,\frac{x}{2}} \sum\limits_{k=1}^{n} [cos(\frac{2k-1}{2})x - cos(\frac{2k+1}{2})x] = \frac{1}{2sin\,\frac{x}{2}}[cos\frac{x}{2} - cos(\frac{2n+1}{2})x] \leq |\frac{1}{sin\frac{x}{2}}|$ is bounded and $\frac{1}{n\pi}$ decreases monotonically to 0, by Dirichlet's test, we know that the Fourier series converges at every point x.

If $a = -\pi$ and $b = \pi$, the Fourier series is 1. $\square$