# Homework 13 Solution

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**Chapter 7. Ex.1** *Let $f$ be a function on the circle. For each $N \leq 1$ the discrete Fourier coefficients of $f$ are defined by*

$$a_N(n) = \frac{1}{N}\sum_{k=1}^{N} f(e^{2\pi ik/N})e^{-2\pi ikn/N}, \ \textit{for } n \in \mathbb{Z}.$$

*We also let*

$$a(n) = \int_0^1 f(e^{2\pi ix})e^{-2\pi inx}dx$$

*denote the ordinary Fourier coefficients of $f$.*
*(a) Show that $a_N(n) = a_N(n+N)$.*
*(b) Prove that if $f$ is continuous, then $a_N(n) \to a(n)$ as $N \to \infty$.*

**Proof.** (a) $a_N(n+N) = \frac{1}{N}\sum\limits_{k=1}^{N} f(e^{\frac{2\pi ik}{N}})e^{-\frac{2\pi ik(n+N)}{N}} = \frac{1}{N}\sum\limits_{k=1}^{N} f(e^{\frac{2\pi ik}{N}})e^{-\frac{2\pi ikn}{N}} = a_N(n)$, since $e^{-2\pi i} = 1$.
(b) Note that $a_N(n)$ is the Riemann sum of the function $f(e^{2\pi ix})e^{-2\pi inx}$ with the partition $\frac{k}{N}, k = 0, 1, ..., N$ of $[0,1]$.
Since $f$ is continuous, $f(e^{2\pi ix})e^{-2\pi inx}$ is integrable in $[0,1]$, yielding that $a_N(n) \to a(n)$ as $N \to \infty$. □

**Chapter 7. Ex.3** *By a similar method, show that if $f$ is a $C^2$ function on the circle, then*

$$|a_N(n) \leq \frac{c}{|n|^2}, \ \textit{whenever } 0 < |n| \leq \frac{N}{2}.$$

*As a result, prove the inversion formula for $f \in C^2$,*

$$f(e^{2\pi ix}) = \sum_{n=-\infty}^{\infty} a(n)e^{2\pi inx}$$

*from its finite version.*

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**Proof.** First we note that

$$\begin{aligned}
|a_N(n)(e^{\frac{2\pi iln}{N}}+e^{-\frac{2\pi iln}{N}}-2)| &= |\frac{1}{N}\sum_{k=1}^{N} f(e^{\frac{2\pi ik}{N}})e^{-\frac{2\pi ik}{N}}(e^{\frac{2\pi iln}{N}}+e^{-\frac{2\pi iln}{N}}-2)| \\
&= |\frac{1}{N}\sum_{k=1}^{N}[f(e^{\frac{2\pi i(k+l)}{N}})+f(e^{\frac{2\pi i(k-l)}{N}})-2f(e^{\frac{2\pi ik}{N}})]e^{-\frac{2\pi ikn}{N}}| \\
&\le \frac{1}{N}\sum_{k=1}^{N}|f'(\xi_1)(e^{\frac{2\pi i(k+l)}{N}}-e^{\frac{2\pi ik}{N}})-f'(\xi_2)(e^{\frac{2\pi ik}{N}}-e^{\frac{2\pi i(k-l)}{N}})| \\
&\le M|1-e^{-\frac{2\pi il}{N}}|\cdot|1-e^{-\frac{4\pi il}{N}}|,
\end{aligned} \tag{1}$$

where $\xi_1 \in (e^{\frac{2\pi ik}{N}}, e^{\frac{2\pi i(k+l)}{N}})$, $\xi_2 \in (e^{\frac{2\pi i(k-l)}{N}}, e^{\frac{2\pi ik}{N}})$, and $M = \max f''(x)$.
Then we choose $l$ such that $|\frac{ln}{N}-\frac{1}{2}| \le \frac{1}{4}$, i.e., $\frac{1}{4} \le \frac{ln}{N} \le \frac{3}{4}$.
Hence $|1-e^{-\frac{2\pi il}{N}}| \le |\frac{2\pi l}{N}| = \frac{2\pi}{|n|}|\frac{nl}{N}| \le \frac{3\pi}{2|n|}$, $|1-e^{-\frac{4\pi il}{N}}| \le |\frac{4\pi l}{N}| \le \frac{3\pi}{|n|}$.
Meanwhile, since $\frac{\pi}{2} \le \frac{2\pi ln}{N} \le \frac{3\pi}{2}$, we know that $|e^{\frac{2\pi iln}{N}}+e^{-\frac{2\pi iln}{N}}-2| \ge 2$.
Therefore, $a_N(n) \le \frac{9\pi^2 M}{4n^2}$.
To obtain the inversion formula, we assume that $N$ is odd. In fact, if $N$ is even, the following summation will be $\sum\limits_{n=-\frac{N}{2}}^{\frac{N}{2}-1}$. Consider

$$\begin{aligned}
\sum_{|n|<\frac{N}{2}} a_N(n)e^{\frac{2\pi ikn}{N}} &= \sum_{|n|<\frac{N}{2}}(\frac{1}{N}\sum_{j=1}^{N} f(e^{\frac{2\pi ij}{N}})e^{-\frac{2\pi ijn}{N}})e^{\frac{2\pi ikn}{N}} \\
&= \frac{1}{N}\sum_{j=1}^{N} f(e^{\frac{2\pi ij}{N}})\sum_{|n|<\frac{N}{2}} e^{\frac{2\pi i(k-j)n}{N}} \\
&= f(e^{\frac{2\pi ik}{N}}),
\end{aligned} \tag{2}$$

since $\sum\limits_{|n|<\frac{N}{2}} e^{\frac{2\pi i(k-j)n}{N}} = 0$ if $j \ne k$ and is equal to $N$ if $j = k$.
Note that when $N$ is large, we can choose $k$ properly such that $|x-\frac{k}{N}| < \epsilon$ for any $\epsilon > 0$.
Since $|a_N(n)| \le \frac{c}{n^2}$ whenever $0 < |n| \le \frac{N}{2}$, the series $\sum\limits_{n=-\infty}^{\infty} a_N(n)e^{\frac{2\pi ikn}{N}}$ converges absolutely and uniformly on the circle as $N \to \infty$.
Hence we can also change the order of summation with limit and obtain that

$$f(e^{2\pi ix}) = \lim_{N\to\infty} f(e^{\frac{2\pi ik}{N}}) = \lim_{N\to\infty}\sum_{|n|<\frac{N}{2}} a_N(n)e^{\frac{2\pi ikn}{N}} = \sum_{n=-\infty}^{\infty}\lim_{N\to\infty} a_N(n)e^{\frac{2\pi ikn}{N}} = \sum_{n=-\infty}^{\infty} a(n)e^{2\pi inx},$$

where we use the fact that $a_N(n) \to a(n)$ as $N \to \infty$ by Exercise 1. $\square$
**Remark:** In reality, one can directly prove the inversion formula without referring to its finite

version.
Since $|a_N(n)| \leq \frac{c}{n^2}$ whenever $0 < |n| \leq \frac{N}{2}$ and $a_N(n) \to a(n)$ as $N \to \infty$, we also have $|a(n)| \leq \frac{c}{n^2}$ for all $n \in \mathbb{Z}$.
Meanwhile, $a(n)$ is the Fourier coefficients of $f(e^{2\pi inx})$. By Corollary 2.4 in Chapter 2, the result follows.

**Chapter 7. Ex.10** *A group $G$ is **cyclic** if there exists $g \in G$ that generates all of $G$, that is, if any element in $G$ can be written as $g^n$ for some $n \in \mathbb{Z}$. Prove that a finite abelian group is cyclic if and only if it is isomorphic to $\mathbb{Z}(N)$ for some $N$.*

**Proof.** ($\Rightarrow$) If $G$ is a finite cyclic group, we denote its generator by $g \in G$ and let $N$ be the smallest positive integer such that $a^N = e$, where $e$ is the identity of $G$.
If $s \in \mathbb{Z}$ and $s = Nq + r$ for $0 \leq r < N$ by Euclidean Division Theorem, then $g^s = g^{Nq+r} = (g^N)^q g^r = e^q g^r = g^r$.
If $0 < k < h < N$ and $g^k = g^h$, then $g^{h-k} = e$ and $0 < h - k < n$, contradicting our choice of $N$. Thus the elements

$$g^0 = e, g, g^2, ..., g^{N-1}$$

are all distinct and comprise all elements of $G$.
This means that the order of $G$ is $N$ and we can construct a well-defined and bijective map $\phi : G \to \mathbb{Z}(N)$ given by $\phi(g^i) = i$ for $i = 0, 1, 2, ..., n-1$. Because $g^N = e$, we see that $g^i g^j = g^k$ where $k = (i + j)\,(mod\, N)$. Thus

$$\phi(g^i g^j) = (i + j)\,(mod\, N) = (\phi(g^i) + \phi(g^j))\,(mod\, N),$$

showing that $\phi$ is an isomorphism.
($\Leftarrow$) If $G$ is isomorphic to $\mathbb{Z}(N)$, then we denote the isomorphism by $\varphi : \mathbb{Z}(N) \to G$.
Then $\varphi(0)$ is the identity of $G$ and $\varphi(1)$ is the generator of $G$, since the order of $G$ is $N$ and the elements in $G$ is of the form $\varphi(n) = n \cdot \varphi(1)$. □

**Chapter 7. Ex.11** *Write down the multiplicative tables for the groups $\mathbb{Z}^*(3), \mathbb{Z}^*(4), \mathbb{Z}^*(5), \mathbb{Z}^*(6), \mathbb{Z}^*(8)$, and $\mathbb{Z}^*(9)$. Which of these groups are cyclic?*

**Proof.** The multiplicative tables for the groups $\mathbb{Z}^*(3), \mathbb{Z}^*(4), \mathbb{Z}^*(5), \mathbb{Z}^*(6), \mathbb{Z}^*(8)$, and $\mathbb{Z}^*(9)$ are

| $\mathbb{Z}^*(3)$ | 1 | 2 |
|---|---|---|
| 1 | 1 | 2 |
| 2 | 2 | 1 |

| $\mathbb{Z}^*(4)$ | 1 | 3 |
|---|---|---|
| 1 | 1 | 3 |
| 3 | 3 | 1 |

| $\mathbb{Z}^*(5)$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 |
| 2 | 2 | 4 | 1 | 3 |
| 3 | 3 | 1 | 4 | 2 |
| 4 | 4 | 3 | 2 | 1 |

| $\mathbb{Z}^*(6)$ | 1 | 5 |
|---|---|---|
| 1 | 1 | 5 |
| 5 | 5 | 1 |

| $\mathbb{Z}^*(8)$ | 1 | 3 | 5 | 7 |
|---|---|---|---|---|
| 1 | 1 | 3 | 5 | 7 |
| 3 | 3 | 1 | 7 | 5 |
| 5 | 5 | 7 | 1 | 3 |
| 7 | 7 | 5 | 3 | 1 |

| $\mathbb{Z}^*(9)$ | 1 | 2 | 4 | 5 | 7 | 8 |
|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 4 | 5 | 7 | 8 |
| 2 | 2 | 4 | 8 | 1 | 5 | 7 |
| 4 | 4 | 8 | 7 | 2 | 1 | 5 |
| 5 | 5 | 1 | 2 | 7 | 8 | 4 |
| 7 | 7 | 5 | 1 | 8 | 4 | 2 |
| 8 | 8 | 7 | 5 | 4 | 2 | 1 |

And the groups $\mathbb{Z}^*(3), \mathbb{Z}^*(4), \mathbb{Z}^*(5), \mathbb{Z}^*(6)$, and $\mathbb{Z}^*(9)$ are cyclic, whose generators are $2, 3, 3, 5, 2$, respectively. $\square$