Homework 12 Solution

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Chapter 5. Ex.21 *Suppose that f is continuous on $\mathbb{R}$. Show that f and $\hat{f}$ cannot both be compactly supported unless $f = 0$. This can be viewed in the same spirit as the uncertainty principle.*

Proof. Assume that f is supported in $[0, \frac{1}{2}]$. We expand f in a Fourier series in the interval $[0, 1]$, i.e., $f(x) \sim \sum\limits_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi inx}$, where $\hat{f}(n) = \int_0^1 f(x)e^{-2\pi inx}dx$.

By Fejér theorem, we know that $f(x) = \lim\limits_{N\to\infty} \sigma_N(f)(x) = \lim\limits_{N\to\infty} \frac{\sum\limits_{k=0}^{N-1} S_k(f)(x)}{N}$.
If we suppose that $\hat{f}$ is also compactly supported, then there exists an $M > 0$ such that $\hat{f}(x) = 0$ when $|x| \geq M$.
Thus, the preceding *Cesàro* mean is just a trigonometric polynomial.
However, a trigonometric polynomial of degree N has at most $2N$ roots in the periodic interval (*finite roots*), contradicting to the fact that $f(x) = 0$ when $x \in (\frac{1}{2}, 1)$ (*infinite roots*).
Hence f and $\hat{f}$ cannot both be compactly supported unless $f = 0$. $\square$

Chapter 5. Ex.22 *The heuristic assertion stated before Theorem 4.1 can be made precise as follows. If F is a function on $\mathbb{R}$, then we say that the preponderance of its mass is contained in an interval I (centered at the origin) if*

$$\int_I x^2|F(x)|^2dx \geq \frac{1}{2}\int_{\mathbb{R}} x^2|F(x)|^2dx. \tag{1}$$

Now suppose $f \in \mathcal{S}$, and (1) holds with $F = f$ and $I = I_1$; also with $F = \hat{f}$ and $I = I_2$. Then if L_j denotes the length of I_j, we have

$$L_1L_2 \geq \frac{1}{2\pi}.$$

A similar conclusion holds if the intervals are not necessarily centered at the origin.

Proof. Given the conditions, we have $\frac{1}{2}\int_{\mathbb{R}} x^2|f(x)|^2dx \leq \int_{I_1} x^2|f(x)|^2dx \leq (\frac{L_1}{2})^2\int_{\mathbb{R}} |f(x)|^2dx$ and $\frac{1}{2}\int_{\mathbb{R}} x^2|\hat{f}(x)|^2dx \leq \int_{I_2} x^2|\hat{f}(x)|^2dx \leq (\frac{L_2}{2})^2\int_{\mathbb{R}} |\hat{f}(x)|^2dx$, where I_1, I_2 are centered at the

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origin. Meanwhile,

$$\begin{aligned}\int_{\mathbb{R}} |f(x)|^2 dx &= \int_{\mathbb{R}} x(f'(x)\overline{f(x)} + f(x)\overline{f'(x)})dx \\ &\le 2\int_{\mathbb{R}} |x| \cdot |f(x)| \cdot |f'(x)| dx \\ &\le 2(\int_{\mathbb{R}} |x|^2 |f(x)|^2 dx)^{\frac{1}{2}} (\int_{\mathbb{R}} |f'(x)|^2 dx)^{\frac{1}{2}} \\ &= 2(\int_{\mathbb{R}} |x|^2 |f(x)|^2 dx)^{\frac{1}{2}} (4\pi^2 \int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 d\xi)^{\frac{1}{2}},\end{aligned} \tag{2}$$

where we use the Cauchy-Schwarz inequality, properties of the Fourier transform, and the Plancherel formula in $\mathcal{S}(\mathbb{R})$.
Therefore, we obtain that

$$\begin{aligned}(\int_{\mathbb{R}} |f(x)|^2 dx)^2 &\le 16\pi^2 (\int_{\mathbb{R}} |x|^2 |f(x)|^2 dx))(\int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 d\xi) \\ &\le 4\pi^2 L_1^2 L_2^2 (\int_{\mathbb{R}} |f(x)|^2 dx)(\int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\xi).\end{aligned} \tag{3}$$

Again by the Plancherel formula, we have $\int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\xi$ and the result follows from the preceding inequality. $\square$