Homework 11 Solution

Yikun Zhang[1]

Chapter 5. Ex.11 *Suppose that u is the solution to the heat equation given by $u = f * \mathcal{H}_t$ where $f \in \mathcal{S}(\mathbb{R})$. If we also set $u(x,0) = f(x)$, prove that u is continuous on the closure of the upper half-plane, and vanishes at infinity, that is,*

$$u(x,t) \to 0 \text{ as } |x| + t \to \infty.$$

Proof. By Theorem 2.1 in Chapter 5, we know that u is continuous on the closure of the upper half-plane.
By definition, we have $u(x,t) = \int_{-\infty}^{\infty} f(x-y) \frac{1}{\sqrt{4\pi t}} e^{-\frac{y^2}{4t}} dy$.
Hence, on one hand, since $e^{-\frac{y^2}{4t}} \leq 1$ and $f \in \mathcal{S}(\mathbb{R})$,

$$|u(x,t)| \leq \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(x-y) dy \leq \frac{C}{\sqrt{t}},$$

where C is a constant.
On the other hand, since f is rapidly decreasing, $|f(x-y)| \leq \frac{C_N}{(1+|x-y|)^N} \leq \frac{4C_N}{(1+|x|)^N}$ when $|y| \leq \frac{|x|}{2}$ and therefore

$$|u(x,t)| \leq \frac{C_N}{(1+|x|)^N} \int_{-\infty}^{\infty} \mathcal{H}_t(y) dy + Ct^{-\frac{1}{2}} e^{-\frac{x^2}{16t}} \int_{|y| \geq \frac{|x|}{2}} |f(x-y)| dy \leq \frac{C}{1+|x|^2} + Ct^{-\frac{1}{2}} e^{-\frac{cx^2}{t}}.$$

Thus, as $|x| + t \to \infty$, in the case when $|x| \leq t$, $|u(x,t)| \leq \frac{C}{\sqrt{t}} \to 0$ because $t \to \infty$;
otherwise, if $|x| > t$, $|u(x,t)| \leq \frac{C}{1+|x|^2} + Ct^{-\frac{1}{2}} e^{-\frac{cx^2}{t}} \to 0$ because $|x| \to \infty$. □

Chapter 5. Ex.14 *Prove that the periodization of the Fejér kernel $\mathcal{F}_N$ on the real line is equal to the Fejér kernel for periodic functions of period 1. In other words,*

$$\sum_{n=-\infty}^{\infty} \mathcal{F}_N(x+n) = F_N(x),$$

when $N \geq 1$ is an integer, and where

$$F_N(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x} = \frac{1}{N} \frac{\sin^2(N\pi x)}{\sin^2(\pi x)}.$$

[1] School of Mathematics, Sun Yat-sen University

Proof. *Method 1*: Consider

$$\hat{f}(\xi) = \begin{cases} 1 - \frac{|\xi|}{N} & \text{if } |\xi| \leq N, \\ 0 & \text{if } |\xi| > N. \end{cases}$$

Obviously, $\hat{f}$ is at least of moderate decrease, since $\hat{f}$ is continuous and supported in $[-N, N]$. Applying the Fourier inversion formula, we know that $f(x) = \mathcal{F}_N(x) = \int_{-N}^{N}(1 - \frac{|\xi|}{N})e^{2\pi i\xi x}d\xi$, where $\mathcal{F}_N$ is the Fejér kernel on the real line.
By the Poisson summation formula, we have

$$\sum_{n=-\infty}^{\infty} \mathcal{F}_N(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi inx} = \sum_{n=-N}^{N}(1 - \frac{|n|}{N})e^{2\pi inx} = F_N(x). \qquad \square$$

Method 2: Note that $N \geq 1$ is an integer. Hence, $\sum\limits_{n=-\infty}^{\infty} \mathcal{F}_N(x+n) = \sum\limits_{n=-\infty}^{\infty} N[\frac{sin(\pi(x+n)N)}{\pi(x+n)N}]^2 = \sum\limits_{n=-\infty}^{\infty} \frac{1}{N\pi^2} \cdot \frac{sin^2(\pi Nx)}{(x+n)^2} = \frac{1}{N} \cdot \frac{sin^2(\pi Nx)}{sin^2(\pi x)}$, where we use the formula for the Fejér kernel in Exercise 9 and the result of Exercise 15 (a). $\square$

Chapter 5. Ex.15 *This exercise provides another example of periodization.*
(a) Apply the Poisson summation formula to the function g, where

$$g(x) = \begin{cases} 1 - |x| & \textit{if } |x| \leq 1, \\ 0 & \textit{otherwise.} \end{cases}$$

and $\hat{g}(\xi) = (\frac{sin\,\pi\xi}{\pi\xi})^2$ to obtain

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)^2} = \frac{\pi^2}{(sin\,\pi\alpha)}$$

whenever α is real, but not equal to an integer.
(b) Prove as a consequence that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)} = \frac{\pi}{\tan \pi\alpha} \tag{1}$$

whenever α is real but not equal to an integer.

Proof. (a) By direct calculations, we know that $\hat{g}(\xi) = \int_{-\infty}^{\infty} g(x)e^{-2\pi ix\xi}dx = (\frac{\sin\pi\xi}{\pi\xi})^2$.
Thanks to the bijective relation of the Fourier transform, we can regard g as the Fourier transform of $\hat{g}$, where both g and $\hat{g}$ are of moderate decrease.
Therefore, by the Poisson summation formula, we have $\sum\limits_{n=-\infty}^{\infty} \frac{\sin^2[\pi(n+\alpha)]}{\pi^2(n+\alpha)^2} = \sum\limits_{n=-\infty}^{\infty} g(n)e^{2\pi in\alpha} = 1$, yielding that $\sum\limits_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)^2} = \frac{\pi^2}{(\sin\pi\alpha)^2}$. $\square$

(b) Assume that $0 < \alpha < 1$. For any $x \in [0,1]$, $\sum\limits_{n\neq 0} \frac{1}{(n+x)^2}$ converges absolutely and uniformly. Therefore,

$$\int_0^\alpha \sum_{n\neq 0} \frac{1}{(n+x)^2} dx = \sum_{n\neq 0} [\int_0^\alpha \frac{dx}{(-n+x)^2} + \int_0^\alpha \frac{dx}{(n+x)^2}] = -(\frac{1}{-n+\alpha} + \frac{1}{n+\alpha}).$$

Since $\lim\limits_{x\to 0}[\frac{\pi^2}{(\sin \pi x)^2} - \frac{1}{x^2}] = \lim\limits_{x\to 0} \frac{\pi^2 x^2 - [\pi x - (\pi x)^3/3! + \cdots]^2}{x^2[\pi x - (\pi x)^3/3! + \cdots]^2} = \lim\limits_{x\to 0} \frac{\pi^4 x^4/3 + o(x^4)}{\pi^2 x^4 + o(x^4)} = \frac{\pi^2}{3}$ and $\lim\limits_{x\to 0}(-\frac{\pi}{\tan \pi x} + \frac{1}{x}] = 0$ also via Taylor expansions, $\int_0^\alpha [\frac{\pi^2}{\sin^2(\pi x)} - \frac{1}{x^2}]dx = -\frac{\pi}{\tan \pi\alpha} + \frac{1}{\alpha}$ makes sense.

Thus, by integrating out x for $\sum\limits_{n=-\infty}^{\infty} \frac{1}{(n+x)^2} = \frac{\pi^2}{(\sin \pi x)^2}$ on $[0,\alpha]$, we have $\frac{1}{\alpha} + \sum_{n=1}^\infty (\frac{1}{-n+\alpha} + \frac{1}{n+\alpha}) = \frac{\pi}{\tan \pi\alpha}$. In the case when $\alpha = \frac{1}{2}$, the right hand side of (1) is 0 and its left hand side is $2 + \sum\limits_{n=1}^{\infty}(\frac{2}{2n+1} - \frac{2}{2n-1})$, whose sum is also 0.

Finally, for any $\alpha_1 \in \mathbb{R}$, we can write it into $\alpha_1 = [\alpha_1] + \alpha$, where $\alpha \in (0,1)$ and $[\alpha_1]$ is the least integer less than or equal to α_1. Then $\sum\limits_{n=-\infty}^{\infty} \frac{1}{(n+\alpha_1)} = \sum\limits_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)} = \frac{\pi}{\tan \pi\alpha} = \frac{\pi}{\tan \pi\alpha_1}$. $\square$