Efficient Inference on High-Dimensional Linear Models With Missing Outcomes

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Joint Work with *Alexander Giessing*[2] and *Yen-Chi Chen*[1]

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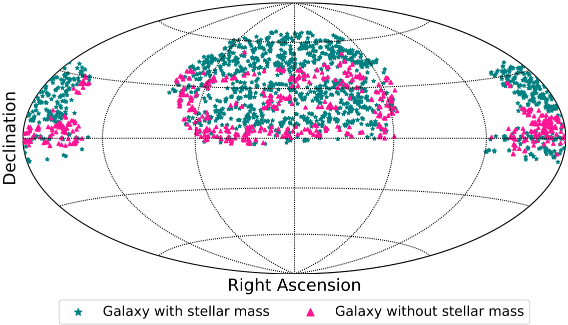
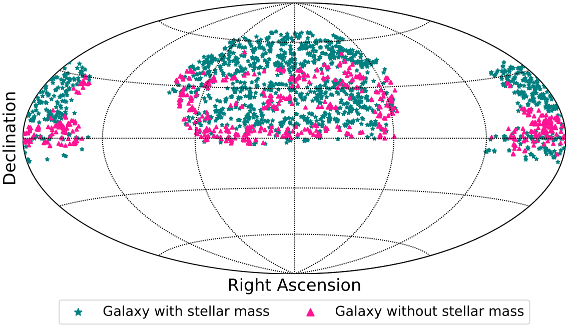


Observed galaxies on the high redshift slice $0.4 \sim 0.401$.

▶ **Notes:** Sloan Digital Sky Survey (SDSS) observes millions of galaxies, but some (estimated) galactic stellar masses are missing in the associated value-added catalog (Comparat et al., 2017).



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- **Scientific Question:**

How can we quantify the uncertainty of the (estimated) stellar mass of a newly observed galaxy based on the spectroscopic and photometric properties?

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- **Reasons for Missingness:**
  - Limiting usage of the observational run in SDSS for galaxy targets;
  - Potential data contamination;
  - Misclassification of galaxies as stars.
- **Statistical Problem:**

How can we conduct valid and efficient inference on the regression function despite missing outcomes?

Model Assumptions

1. **Linearity:** The data $\{(Y_i, R_i, X_i)\}_{i=1}^n$ are i.i.d. observations from a sparse linear model

$$Y = X^T\beta_0 + \epsilon \quad \text{with} \quad \mathrm{E}\left(\epsilon \middle| X\right) = 0 \quad \text{and} \quad \mathrm{E}\left(\epsilon^2 \middle| X\right) = \sigma_\epsilon^2,$$

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Our method can be generalized to handle heteroscedastic errors.

2. **Missing At Random (MAR):** $Y_i \perp\!\!\!\perp R_i | X_i$ for $i = 1, ..., n$.

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1. **Fully Observed Outcomes:** Debiased Lasso (Zhang and Zhang, 2014; van de Geer et al., 2014; Javanmard and Montanari, 2014):

$$\widehat{\beta}^{\text{debias}} = \widehat{\beta}_\lambda + \frac{1}{n}\widehat{\Theta}\sum_{i=1}^{n} X_i(Y_i - X_i^T\widehat{\beta}_\lambda),$$

   - $\widehat{\beta}_\lambda$ is a Lasso solution under the regularization parameter $\lambda > 0$;
   - $\widehat{\Theta} \in \mathbb{R}^{d\times d}$ is an approximation to the matrix inverse $\left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i^T\right)^{-1}$.

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2. **MAR Outcomes:** M-estimation framework with a Lasso-type debiased and doubly robust estimator (Chakrabortty et al., 2019).

Our Contributions

- **Drawbacks of Existing Approaches:** Inference on $\beta_0 \in \mathbb{R}^d$.

1. Need to compute a $d \times d$ debiasing matrix $\widehat{\Theta}$.

2. Require sample splitting or cross fitting for valid inference.

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- **Our Focus:** Inference on $m_0(x) = x^T \beta_0$.

- *Computational efficiency:* Our debiasing program is convex and only needs to solve for an n-dimensional weight vector.
- *Statistical efficiency*: Our estimator is semi-parametrically efficient among all asymptotically linear estimators.

Methodology and Asymptotic Theory

Heuristics From Debiased Lasso

- The debiased Lasso estimator on the complete-case data is given by

$$\widehat{\beta}^{\text{debias}} = \widehat{\beta}_\lambda + \frac{1}{n}\sum_{i=1}^{n} R_i \widehat{\Theta} X_i \left(Y_i - X_i^T \widehat{\beta}_\lambda\right).$$

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► **Issue:** This naive estimator may not be asymptotically normal in general (van de Geer et al., 2014; Javanmard and Montanari, 2014)!

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- **Idea:** Introduce a weight vector $\boldsymbol{w} = (w_1, ..., w_n)^T \in \mathbb{R}^n$ to replace

$$\frac{1}{\sqrt{n}} x^T \widehat{\Theta} X_i \implies w_i \quad \text{for} \quad i = 1, ..., n$$

and formulate a generic debiased estimator

$$\widehat{m}^{\text{debias}}(x; \boldsymbol{w}) = x^T \widehat{\beta} + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} w_i R_i \left(Y_i - X_i^T \widehat{\beta} \right). \tag{1}$$

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- **Question:** How do we estimate the weight vector $\boldsymbol{w} = (w_1, ..., w_n)^T$?

Conditional Mean Squared Error Decomposition

The conditional mean squared error of $\sqrt{n}\, m^{\text{debias}}(x; \boldsymbol{w})$ is

$$\mathrm{E}\left[\left(\sqrt{n}\, m^{\text{debias}}(x; \boldsymbol{w}) - \sqrt{n}\, m_0(x)\right)^2 \Big| X_1, ..., X_n\right]$$

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&\mathrm{E}\left[\left(\sqrt{n}\, m^{\text{debias}}(x;\boldsymbol{w}) - \sqrt{n}\, m_0(x)\right)^2 \middle| X_1, \dots, X_n\right] \\
&= \underbrace{\sigma_\epsilon^2 \sum_{i=1}^n w_i^2 \pi_i}_{\text{Main Conditional Variance}} + \underbrace{\left[\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n w_i \pi_i X_i - x\right)^T \sqrt{n}\left(\beta_0 - \beta\right)\right]^2}_{\text{Conditional Bias}} \\
&\quad \underbrace{+ \left(\beta_0 - \beta\right)^T \left[\sum_{i=1}^n w_i^2 \pi_i \left(1 - \pi_i\right) X_i X_i^T\right] \left(\beta_0 - \beta\right)}_{\text{Asymptotically Negligible Conditional Variance}}.
\end{aligned}
$$

- **Notes:** $\pi_i := \mathrm{P}(R_i = 1 | X_i)$ is the propensity score under the MAR condition.

Bias-Variance Trade-off Optimization

$$\mathrm{E}\left[\left(\sqrt{n}\, m^{\text{debias}}(x;\boldsymbol{w}) - \sqrt{n}\, m_0(x)\right)^2 \middle| X_1, ..., X_n\right]$$

$$\asymp \underbrace{\sigma_\epsilon^2 \sum_{i=1}^n w_i^2 \pi_i}_{\text{Main Conditional Variance}} + \underbrace{\left[\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n w_i \pi_i X_i - x\right)^T \sqrt{n}\,(\beta_0 - \beta)\right]^2}_{\text{Conditional Bias}}.$$

- By Hölder's inequality,

$$\text{“Conditional Bias”} \leq \left[\left\|\frac{1}{\sqrt{n}} \sum_{i=1}^n w_i \pi_i X_i - x\right\|_\infty \sqrt{n}\,||\beta_0 - \beta||_1\right]^2.$$

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- We design our debiasing program as:

$$\min_{\boldsymbol{w} \in \mathbb{R}^n} \sum_{i=1}^n w_i^2 \widehat{\pi}_i \quad \text{subject to} \quad \left\|x - \frac{1}{\sqrt{n}} \sum_{i=1}^n w_i \cdot \widehat{\pi}_i \cdot X_i\right\|_\infty \leq \frac{\gamma}{n}.$$

1. Compute the Lasso pilot estimate $\widehat{\beta}_\lambda$ on the complete-case data

$$\widehat{\beta}_\lambda = \mathop{\arg\min}_{\beta \in \mathbb{R}^d} \left[\frac{1}{2n} \sum_{i=1}^n R_i (Y_i - X_i^T \beta)^2 + \lambda \left|\left|\beta\right|\right|_1 \right].$$

Our Debiasing Inference Procedure

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4. Define the debiased estimator for $m_0(x) = x^T \beta$ as:

$$\widehat{m}^{\text{debias}}(x; \widehat{\boldsymbol{w}}) = x^T \widehat{\beta} + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \widehat{w}_i R_i \left(Y_i - X_i^T \widehat{\beta} \right).$$

Our Debiasing Inference Procedure

1. How to select the tuning parameter $\gamma > 0$ for our debiasing program?

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▶ **Answer:** The above two questions can be addressed by the *dual formulation* of our debiasing program!

- **Primal Program:**

$$\min_{\boldsymbol{w}\in\mathbb{R}^n}\left\{\sum_{i=1}^{n}\widehat{\pi}_i w_i^2 : \left\|x-\frac{1}{\sqrt{n}}\sum_{i=1}^{n} w_i\cdot\widehat{\pi}_i\cdot X_i\right\|_\infty \le \frac{\gamma}{n}\right\}.$$

Dual Formulation of Our Debiasing Program

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- **Dual Program:**

$$\min_{\ell\in\mathbb{R}^d} \left\{ \frac{1}{4n} \sum_{i=1}^n \widehat{\pi}_i \left(X_i^T \ell \right)^2 + x^T \ell + \frac{\gamma}{n} \left|\left|\ell\right|\right|_1 \right\}.$$

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- **Primal-Dual Relation:** Under the strong duality,

$$\widehat{w}_i = -\frac{1}{2\sqrt{n}} \cdot X_i^T \widehat{\ell} \quad \text{for} \quad i = 1, ..., n.$$

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- It is an *unconstrained* optimization problem, and $\gamma>0$ can be fine-tuned via cross-validation.

- Primal-dual relation $\widehat{w}_i=-\frac{1}{2\sqrt{n}}\cdot X_i^T\widehat{\ell},\ i=1,...,n$ and dual consistency $\widehat{\ell}\xrightarrow{P}\ell_0$ reveal that

$$\sqrt{n}\left[\widehat{m}^{\text{debias}}(x;\widehat{\boldsymbol{w}})-m_0(x)\right]=\underbrace{-\frac{1}{2\sqrt{n}}\sum_{i=1}^{n}R_i\epsilon_iX_i^T\ell_0}_{\text{i.i.d. sum!}}+\underbrace{\text{“Bias terms”}}_{o_P(1)}.$$

Consistency and Asymptotic Normality

Theorem (Theorem 7 in Zhang et al. 2023)

Under regularity conditions,

$$\sqrt{n}\left[\widehat{m}^{\text{debias}}(x;\widehat{\boldsymbol{w}}) - m_0(x)\right] \xrightarrow{d} \mathcal{N}\left(0,\, \sigma_m^2(x)\right)$$

with $\sigma_m^2(x) = \lim_{n\to\infty} \sigma_\epsilon^2 \cdot x^T \left[\mathrm{E}\left(RXX^T\right)\right]^{-1} x.$

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2. Under regularity conditions (Proposition 8 in Zhang et al. 2023),

$$\widehat{\sigma}_\epsilon^2 \sum_{i=1}^{n} \widehat{\pi}_i \widehat{w}_i^2 \xrightarrow{P} \sigma_\epsilon^2 \cdot x^T \left[\mathrm{E}\left(RXX^T\right)\right]^{-1} x.$$

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Theoretical Insights into Propensity Score Estimation

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- Why don't we need sample splitting or cross fitting for estimating the propensity score by any machine learning method?

▶ **Answer:** Our asymptotic normality result depends on the *in-sample* estimation error r_π of the propensity score:

$$\max_{1\leq i\leq n} |\widehat{\pi}_i - \pi_i| = O_P\left(r_\pi\right) \quad \text{with} \quad \pi_i = \pi(X_i), i = 1, ..., n.$$

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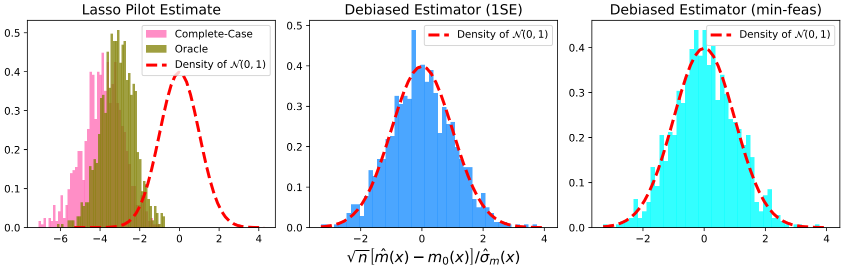
- Our debiased estimator performs even better when the estimated propensity scores on the training data are close to the true ones!!
- This permits the use of complex machine learning methods with high learnability (Steinwart, 2001; Farrell et al., 2021; Gao et al., 2022).

Simulation and Real-World Application



▶ **Effectiveness of Our Debiased Estimator:**

- Correct the bias of the Lasso pilot estimate.
- Asymptotically normal under a wide range of $\gamma > 0$.

▶ **Notes:** Our paper contains comprehensive comparisons with other existing methods.

Results on Galactic Stellar Mass Inference

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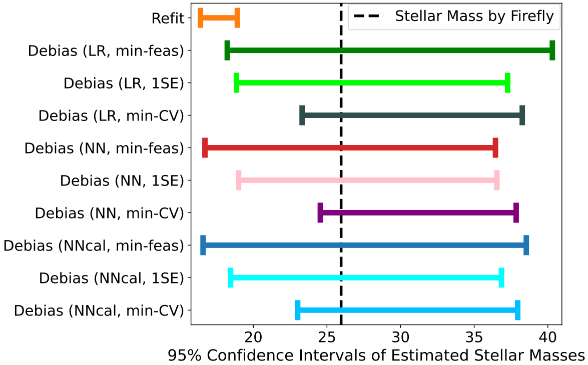
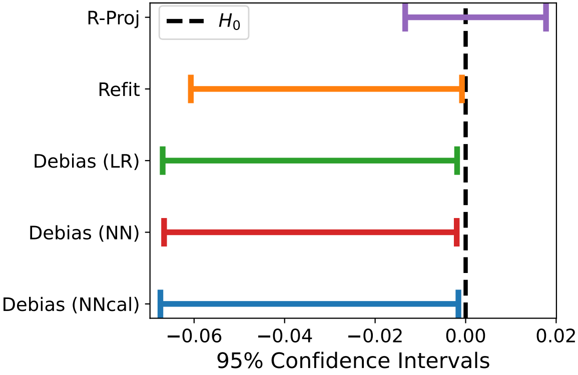


- The 95% confidence intervals by our debiasing methods cover the true stellar mass of a new galaxy.

Results on Galactic Stellar Mass Inference

Is it statistically significant that the stellar mass of a galaxy is negatively correlated with its distance to the nearby cosmic filament structures?



- 95% confidence intervals by our debiasing methods exclude 0 and are all negative.

Summary of the Talk

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More details can be found in

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Python Package: Debias-Infer and R Package: DebiasInfer.

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Thank you!

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Implementation Details of the Proposed Debiasing Method

1. **Lasso pilot estimate:** We adopt the scaled Lasso (Sun and Zhang, 2012) with its universal regularization parameter $\lambda_0 = \sqrt{\frac{2\log d}{n}}$ as the initialization. Specifically, it iteratively updates $\widehat{\beta}(\widetilde{\lambda}), \widehat{\sigma}_\epsilon(\widetilde{\lambda}), \widetilde{\lambda}$ via the jointly convex optimization program:

$$\left(\widehat{\beta}(\widetilde{\lambda}), \widehat{\sigma}_\epsilon(\widetilde{\lambda})\right) = \mathop{\arg\min}_{\beta\in\mathbb{R}^d, \sigma_\epsilon>0} \left[\frac{1}{2n\sigma_\epsilon}\sum_{i=1}^n R_i\left(Y_i - X_i^T\beta\right)^2 + \frac{\sigma_\epsilon}{2} + \widetilde{\lambda}\,||\beta||_1\right].$$

2. **Debiasing program:** We solve the primal program by Python package "`CVXPY`" (Diamond and Boyd, 2016; Agrawal et al., 2018) or R package "`CVXR`" (Fu et al., 2020). For the dual program, we formulate a coordinate descent algorithm (Wright, 2015) as:

$$\left[\widehat{\ell}(x)\right]_j \leftarrow \frac{\mathcal{S}_{\frac{\gamma}{n}}\left(-\frac{1}{2n}\sum_{i=1}^n \widehat{\pi}_i\left(\sum_{k\neq j} X_{ik}X_{jk}\left[\widehat{\ell}(x)\right]_k\right) - x_j\right)}{\frac{1}{2n}\sum_{i=1}^n \widehat{\pi}_i X_{ij}^2} \quad \text{for } j = 1, \dots, d,$$

where $\mathcal{S}_{\frac{\gamma}{n}}(u) = \mathrm{sign}(u)\cdot\left(u - \frac{\gamma}{n}\right)_+$ is the soft-thresholding operator.

One Standard Error (1SE) Rule For Model Selection

- Suppose that we conduct a K-fold cross-validation on a candidate set $\Gamma = \{\gamma_1, ..., \gamma_m\}$ of the tuning parameter.

- For each $\gamma_i \in \Gamma$, we compute the cross-validated risk or error on each fold of the data as:
$$CV_k(\gamma_i), \quad k = 1, ..., K.$$
- For each $\gamma_i \in \Gamma$, we calculate the standard error of $CV_1(\gamma_i), ..., CV_K(\gamma_i)$ as:
$$SD(\gamma_i) = \sqrt{\mathrm{Var}\left(CV_1(\gamma_i), ..., CV_K(\gamma_i)\right)}, \quad SE(\gamma_i) = SD(\gamma_i)/\sqrt{K}.$$
- Let
$$CV(\gamma) = \frac{1}{K}\sum_{k=1}^{K} CV_k(\gamma) \quad \text{and} \quad \widehat{\gamma} = \arg\min_{\gamma \in \Gamma} CV(\gamma).$$
The 1SE rule (Breiman et al., 1984; Chen and Yang, 2021) selects $\gamma_{1SE} \in \Gamma$ with as the one with the smallest $CV(\gamma)$ such that
$$CV(\gamma_{1SE}) \geq CV(\widehat{\gamma}) + SE(\widehat{\gamma}).$$

One Standard Error (1SE) Rule For Model Selection

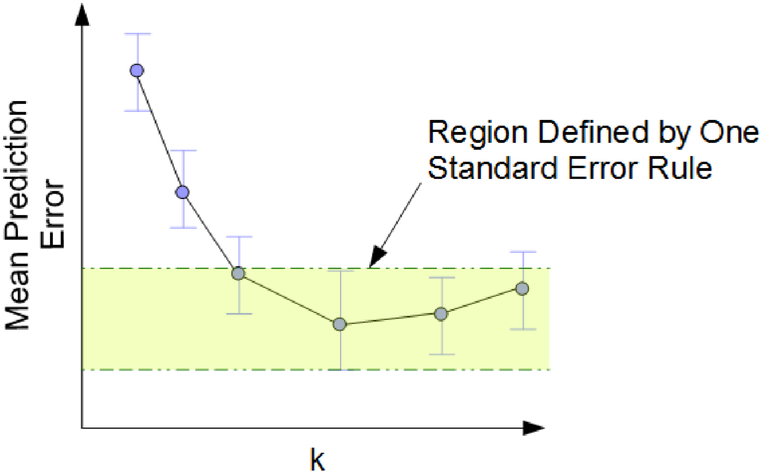


Figure: Illustration of the 1SE rule for selecting the model parameter.

Interpretations From Neyman Near-Orthogonalization

- Consider the regression function $m \equiv m(x) \in \mathbb{R}$ as the main parameter to be inferred and $\beta \in \mathbb{R}^d$ as the high-dimensional nuisance parameter.

- Our generic debiased estimator $m^{\text{debias}}(x, \boldsymbol{w})$ solves the sample-based estimating equation

$$\frac{1}{n}\sum_{i=1}^{n} \Xi_x(Y_i, R_i, X_i; m^{\text{debias}}, \beta) = m^{\text{debias}}(x; \boldsymbol{w}) - x^T\beta - \frac{1}{\sqrt{n}}\sum_{i=1}^{n} w_i \cdot R_i \left(Y_i - X_i^T\beta\right) = 0.$$

- The Neyman near-orthogonalization condition (Chernozhukov et al., 2018) given $\boldsymbol{X} = (X_1, ..., X_n)^T \in \mathbb{R}^{n\times d}$ at $(m_0, \beta_0) = (x^T\beta_0, \beta_0)$ requires

$$\begin{aligned}
&\mathrm{E}\left[\frac{1}{n}\sum_{i=1}^{n} \Xi_x(Y_i, R_i, X_i; m_0, \beta_0)\middle|\boldsymbol{X}\right] = 0, \\
&\sup_{\beta\in\mathcal{T}_n}\left|\left\{\frac{\partial}{\partial\beta}\mathrm{E}\left[\frac{1}{n}\sum_{i=1}^{n} \Xi_x(Y_i, R_i, X_i; m, \beta)\middle|\boldsymbol{X}\right]\Big|_{(m_0,\beta_0)}\right\}^T(\beta - \beta_0)\right| \le \frac{\delta_n}{\sqrt{n}},
\end{aligned} \tag{2}$$

where $\mathcal{T}_n$ is a properly shrinking neighborhood of β_0 and $\delta_n = o(1)$.

- Both conditions in (2) hold true, because for any $\beta \in \mathcal{T}_n$ and some convex set $\mathcal{B}$ containing β_0, we have that

$$
\begin{aligned}
&\left| \left\{ \frac{1}{n} \sum_{i=1}^{n} \frac{\partial}{\partial \beta} \mathrm{E}\left[\Xi_x(Y_i, R_i, X_i; m, \beta) | X \right] \Big|_{(m_0, \beta_0)} \right\}^T (\beta - \beta_0) \right| \\
&= \left| \left[x - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} w_i \cdot \pi(X_i) X_i \right]^T (\beta_0 - \beta) \right| \\
&\text{“} \leq \text{”} \left\| x - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} w_i \cdot \widehat{\pi}_i \cdot X_i \right\|_\infty ||\beta - \beta_0||_1 \quad \text{by Hölder's inequality} \\
&\leq \frac{\gamma}{n} ||\beta - \beta_0||_1 \quad \text{by the box constraint in our debiasing program} \\
&\leq \frac{\delta_n}{\sqrt{n}} \quad \text{by setting } \mathcal{T}_n = \left\{ \beta \in \mathcal{B} \subset \mathbb{R}^d : ||\beta - \beta_0||_1 \leq \frac{\sqrt{n}\delta_n}{\gamma} \right\}.
\end{aligned}
$$

- Our debiasing program optimizes the (estimated) variance among all the estimators satisfying Neyman near-orthogonalization (2).
- (2) also allows our debiasing program to *de-correlate* the Lasso pilot regression from propensity score estimation and weight optimization.

Theoretical Implications of Our Dual Debiasing Program

- **Goal:** Establish the asymptotic normality of our debiased estimator

$$\widehat{m}^{\text{debias}}(x;\widehat{\boldsymbol{w}}) = x^T\widehat{\beta} + \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\widehat{w}_i R_i\left(Y_i - X_i^T\widehat{\beta}\right).$$

- Linearity assumption $Y_i = X_i^T\beta_0 + \epsilon_i$ for $i = 1, ..., n$ implies

$$\sqrt{n}\left[\widehat{m}^{\text{debias}}(x;\widehat{\boldsymbol{w}}) - m_0(x)\right] = \underbrace{\sum_{i=1}^{n}\widehat{w}_i R_i\epsilon_i}_{\text{Not an i.i.d. sum!}} + \left[x - \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\widehat{w}_i R_i X_i\right]^T\sqrt{n}\left(\widehat{\beta} - \beta_0\right),$$

- Dual relation $\widehat{w}_i = -\frac{1}{2\sqrt{n}}\cdot X_i^T\widehat{\ell}$ for $i = 1, ..., n$ and dual consistency $\widehat{\ell} \xrightarrow{P} \ell_0$ reveal that

$$\begin{aligned}\sqrt{n}\left[\widehat{m}^{\text{debias}}(x;\widehat{\boldsymbol{w}}) - m_0(x)\right] &= -\frac{1}{2\sqrt{n}}\sum_{i=1}^{n}R_i\epsilon_i X_i^T\widehat{\ell} + \left[x + \frac{1}{2n}\sum_{i=1}^{n}R_i X_i X_i^T\widehat{\ell}\right]^T\sqrt{n}\left(\beta_0 - \widehat{\beta}\right)\\ &= \underbrace{-\frac{1}{2\sqrt{n}}\sum_{i=1}^{n}R_i\epsilon_i X_i^T\ell_0}_{\text{i.i.d. sum!}} + \underbrace{\text{``Bias terms''}}_{o_P(1)}.\end{aligned}$$

1. The covariate vector $X \in \mathbb{R}^d$ and the noise $\epsilon \in \mathbb{R}$ are sub-Gaussian.

2. There exists a constant $\kappa_R > 0$ such that
$$\inf_{v \in \mathbb{S}^{d-1}} \mathrm{E}\left[R(X^T v)^2\right] \geq \kappa_R^2 \quad \text{with} \quad \mathbb{S}^{d-1} = \left\{x \in \mathbb{R}^d : ||x||_2 = 1\right\}.$$

3. Given any $n \geq 1$ and $\delta \in (0,1)$, there exists $r_\pi \equiv r_\pi(n,\delta) > 0$ such that
$$\mathrm{P}\left(\max_{1 \leq i \leq n} |\widehat{\pi}_i - \pi_i| > r_\pi\right) < \delta \quad \text{with} \quad \pi_i = \pi(X_i), i = 1, ..., n.$$

4. Define the population dual program as:
$$\min_{\ell \in \mathbb{R}^d} \left\{\frac{1}{4}\mathrm{E}\left[R\left(X^T \ell\right)^2\right] + x^T \ell\right\},$$
whose exact solution is $\ell_0(x) = -2\left[\mathrm{E}\left(RXX^T\right)\right]^{-1} x$. We assume that the r_ℓ-approximation $\widetilde{\ell}(x)$ to $\ell_0(x)$ is sparse with $r_\ell \in \left[0, \frac{1}{2}\right]$, *i.e.*,
$$s_\ell(x) = \left|\left|\widetilde{\ell}(x)\right|\right|_0 \ll \min\{n, d\} \;\text{ with }\; \widetilde{\ell}(x) = \arg\min_{u \in \mathbb{R}^d}\left\{||u||_0 : ||u - \ell_0(x)||_2 \leq r_\ell \,||\ell_0(x)||_2\right\}.$$

Methods to be compared:

- **"DL-Jav":** The debiased Lasso by Javanmard and Montanari (2014).
- **"DL-vdG":** The debiased Lasso by van de Geer et al. (2014).
- **"Refit":** Run the regular least-square regression on the support set of the Lasso pilot estimate (Belloni and Chernozhukov, 2013).

Implementation settings of the above methods:

- Complete-case (CC) data $\{(X_i, Y_i, R_i = 1)\}_{i=1}^n$;
- Inverse probability weighted (IPW) data $\left\{\left(\frac{X_i}{\sqrt{\widehat{\pi}_i}}, \frac{Y_i}{\sqrt{\widehat{\pi}_i}}, R_i = 1\right)\right\}_{i=1}^n$;
- Oracle fully observed data (X_i, Y_i) for $i = 1, ..., n$.

Evaluation metrics over 1000 Monte Carlo experiments:

- Average absolute bias $\left|\widehat{m}^{\text{debias}}(x) - m_0(x)\right|$;
- Average coverage and average length of the yielded 95% confidence intervals.

Simulation Results Under Gaussian Noises (I)

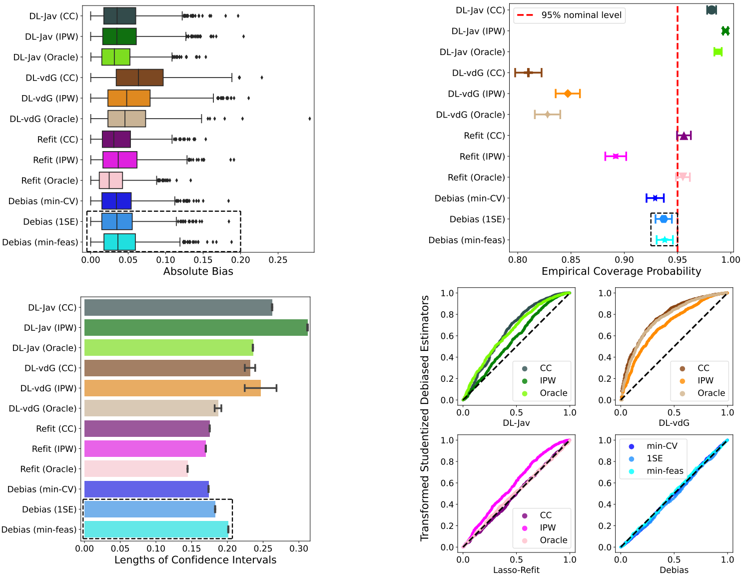


Figure: Sparse β_0^{sp} and sparse $x^{(2)}$ with $X_i \sim \mathcal{N}_d(\mathbf{0}, \Sigma^{\mathrm{cs}}), i = 1, ..., n$.

Simulation Results Under Gaussian Noises (II)

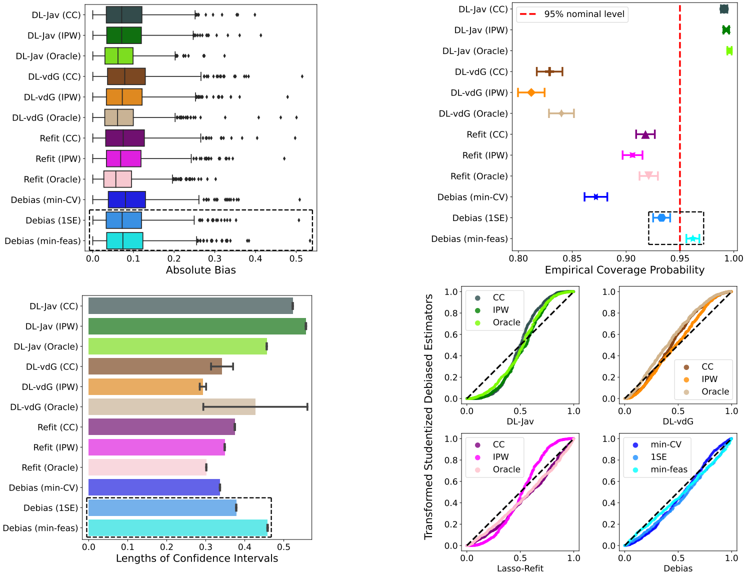


Figure: Pseudo-dense β_0^{pd} and sparse $x^{(2)}$ with $X_i \sim \mathcal{N}_d(\mathbf{0}, \Sigma^{\text{ar}}), i = 1, ..., n$.

Simulation Results Under $\text{Laplace}(0, 1/\sqrt{2})$ Noises

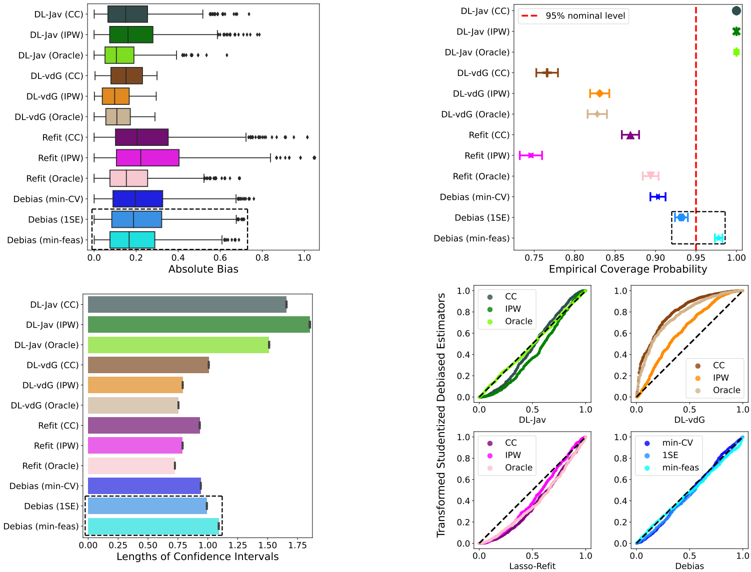


Figure: Dense β_0^{de} and sparse $x^{(2)}$ with $X_i \sim \mathcal{N}_d(\mathbf{0}, \Sigma^{cs}), i = 1, ..., n$.

Simulation Results Under t_2-Distributed Noises

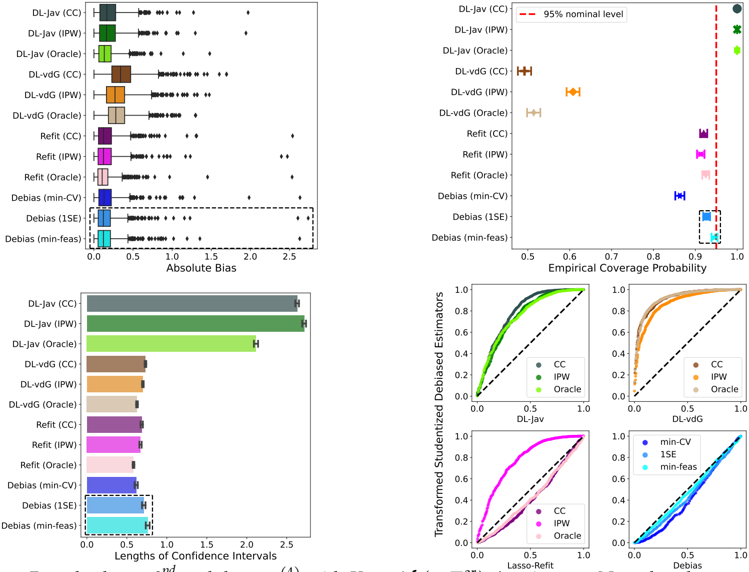


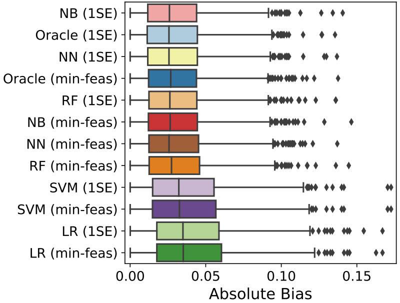
Figure: Pseudo-dense β_0^{pd} and dense $x^{(4)}$ with $X_i \sim \mathcal{N}_d(\mathbf{0}, \Sigma^{\text{ar}}), i = 1, \ldots, n$. Note that the mean-zero t_2 distribution has *infinite* variance.

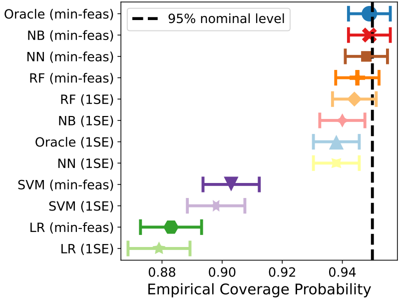
1. True propensity score model: $\mathrm{P}(R_i = 1|X_i) = \Phi\left(-4 + \sum_{k=1}^{K} Z_{ik}\right)$, where $(Z_{i1}, ..., Z_{iK})$ contains all polynomial combinations of the first eight components $X_{i1}, ..., X_{i8}$ of $X_i \in \mathbb{R}^{1000}$ with degrees ≤ 2.

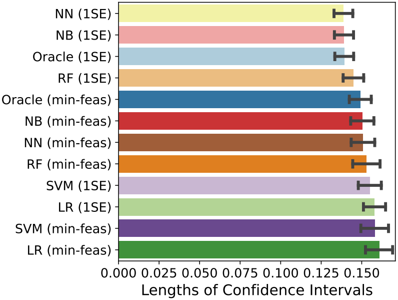
2. Estimate the propensity scores $\pi(X_i), i = 1, ..., n$ by the following nonlinear/nonparametric machine learning methods:
   - **Gaussian Naive Bayes ("NB")**.
   - **Random Forest ("RF"):** 100 trees, bootstrapping samples, and the Gini impurity.
   - **Support Vector Machine ("SVM"):** Gaussian radial basis function.
   - **Neural Network ("NN"):** Two hidden layers of size 80×50 and ReLU $h(x) = \max\{x, 0\}$ as the activation function.

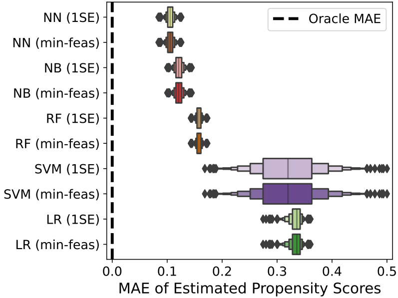
3. Include an extra evaluation metric as the average mean absolute error ("Avg-MAE") for the estimated propensity scores.

Simulation Results With Nonparametric Propensity Scores

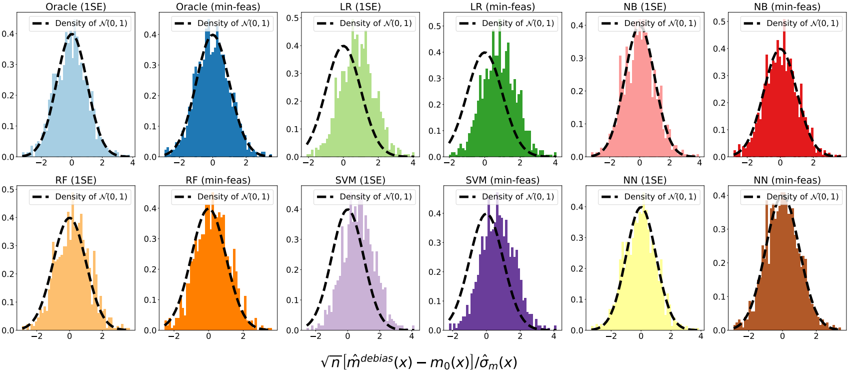








Simulation Results With Nonparametric Propensity Scores



Study Design For Stellar Mass Inference

1. Consider all the observed galaxies by SDSS-IV within a thin redshift slice $0.4 \sim 0.4005$, among which 30.2% of their stellar masses are missing in the Firefly value-added catalog.
2. Fetch their spectroscopic and photometric properties from SDSS-IV DR16 database similar to the input catalog of Chang et al. (2015).
3. Apply feature transformation, remove highly linearly correlated covariates, and generate univariate B-spline base covariates of polynomial order 3 with 40 knots.
4. Incorporate RA, DEC, and the angular diameter distances from the galaxies to the two-dimensional spherical cosmic filaments by Zhang and Chen (2023); Zhang et al. (2022).
5. Control for the confounding effects by including the distances from galaxies to candidate galaxy clusters.

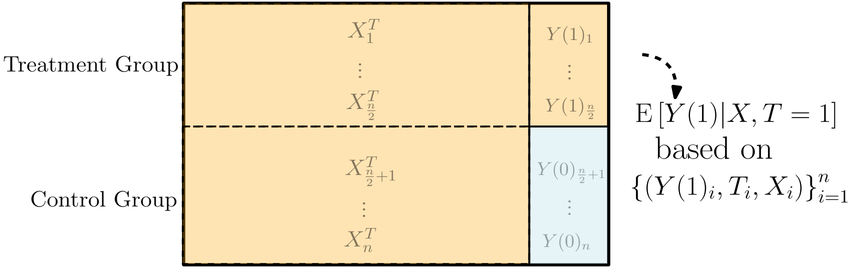
- **Final Dataset:** $n = 1185$ and $d = 1409$.

The observable data in causal inference are

$$\{(\mathbb{Y}_i, T_i, X_i)\}_{i=1}^n \subset \mathbb{R} \times \{0,1\} \times \mathbb{R}^d.$$

- $T_i \in \{0,1\}$ is a binary treatment assignment indicator;
- $\mathbb{Y}_i = T_i \cdot Y(1)_i + (1 - T_i) \cdot Y(0)_i$ with $Y(0), Y(1)$ as potential outcomes.

▶ **Objective:** Conduct valid inference on $\mathrm{E}\left[Y(1)|X, T=1\right]$.



Our debiasing method can be extended to valid inference on the high-dimensional linear average conditional treatment effect (ACTE)

$$\mathrm{E}[Y(1) - Y(0)|X].$$

- The modified debiasing program with tuning parameters $\gamma_1, \gamma_2 > 0$ is

$$\operatorname*{arg\,min}_{\boldsymbol{w}_{(0)}, \boldsymbol{w}_{(1)} \in \mathbb{R}^n} \sum_{i=1}^{n} \left[\widehat{\pi}_i w_{i(1)}^2 + (1 - \widehat{\pi}_i) w_{i(0)}^2 \right]$$

$$\text{s.t.} \quad \left\| x - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} w_{i(1)} \cdot \widehat{\pi}_i \cdot X_i \right\|_{\infty} \leq \frac{\gamma_1}{n} \quad \text{and} \quad \left\| x - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} w_{i(0)} \left(1 - \widehat{\pi}_i\right) X_i \right\|_{\infty} \leq \frac{\gamma_2}{n}.$$

- The extended debiased estimator becomes

$$\widehat{m}^{\text{debias}}(x; \widehat{\boldsymbol{w}}_{(1)}, \widehat{\boldsymbol{w}}_{(0)})$$
$$= x^T \left(\widehat{\beta}_{(1)} - \widehat{\beta}_{(0)} \right) + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\widehat{w}_{i(1)} \cdot T_i \left(\mathbb{Y}_i - X_i^T \widehat{\beta}_{(1)} \right) - \widehat{w}_{i(0)} \cdot (1 - T_i) \left(\mathbb{Y}_i - X_i^T \widehat{\beta}_{(0)} \right) \right].$$

- The efficiency theory for this modified procedure is worth studying!

Potential Application to Causal Inference (I)

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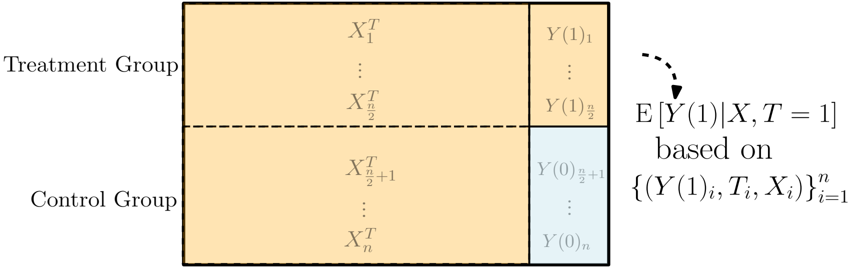

Potential Application to Causal Inference (II)

Our debiasing method can be extended to valid inference on the high-dimensional linear average conditional treatment effect (ACTE)

$$\mathrm{E}[Y(1) - Y(0)|X].$$

- The modified debiasing program with tuning parameters $\gamma_1, \gamma_2 > 0$ is

$$\mathop{\arg\min}_{\boldsymbol{w}_{(0)}, \boldsymbol{w}_{(1)} \in \mathbb{R}^n} \sum_{i=1}^{n} \left[\widehat{\pi}_i w_{i(1)}^2 + (1 - \widehat{\pi}_i) w_{i(0)}^2 \right]$$

$$\text{s.t. } \left\| x - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} w_{i(1)} \cdot \widehat{\pi}_i \cdot X_i \right\|_\infty \leq \frac{\gamma_1}{n} \text{ and } \left\| x - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} w_{i(0)} \left(1 - \widehat{\pi}_i\right) X_i \right\|_\infty \leq \frac{\gamma_2}{n}.$$

- The extended debiased estimator becomes

$$\begin{aligned} &\widehat{m}^{\text{debias}}(x; \widehat{\boldsymbol{w}}_{(1)}, \widehat{\boldsymbol{w}}_{(0)}) \\ &= x^T \left(\widehat{\beta}_{(1)} - \widehat{\beta}_{(0)} \right) + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\widehat{w}_{i(1)} \cdot T_i \left(\mathbb{Y}_i - X_i^T \widehat{\beta}_{(1)} \right) - \widehat{w}_{i(0)} \cdot (1 - T_i) \left(\mathbb{Y}_i - X_i^T \widehat{\beta}_{(0)} \right) \right]. \end{aligned}$$

- The efficiency theory for this modified procedure is worth studying!

The galaxy distribution is distorted along the line of sight due to the peculiar velocities of galaxies, *i.e.*, the so-called *finger-of-god* (Jackson, 1972) and *Kaiser* (Kaiser, 1987) effects.

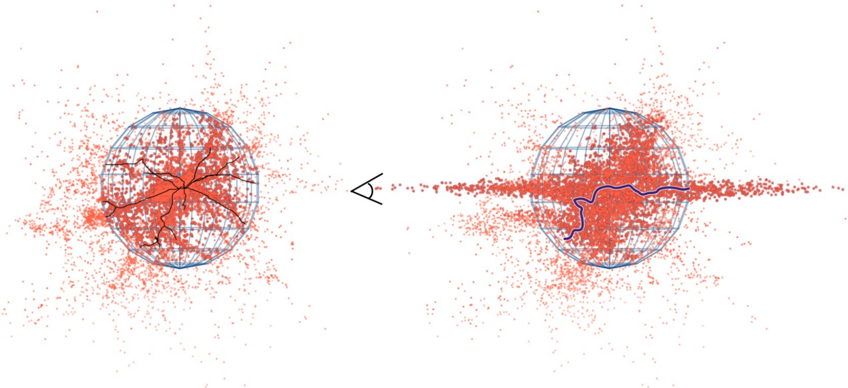

Figure: Redshift distortions along the line of sight (Kuchner et al., 2021).