Efficient Inference on High-Dimensional Linear Models with Missing Outcomes

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Problem of Interest

Statistical inference on the conditional mean $m_0(x) = \mathrm{E}(Y|X=x)$ in the presence of high-dimensional covariates and missing outcomes is of practical importance. How can we conduct this inference efficiently?

The efficiency should come from two aspects:

1. The final estimator of $m_0(x)$ is *semi-parametrically efficient* among a certain class of estimators.
2. The entire inference procedures are *computationally efficient*.

Basic Assumptions: Consider a random sample $\{(Y_i, R_i, X_i)\}_{i=1}^n$ drawn from the joint distribution of $(Y, R, X) \in \mathbb{R} \times \{0,1\} \times \mathbb{R}^d$ satisfying

(a) $Y = X^T\beta_0 + \epsilon$ with $\mathrm{E}(\epsilon|X) = 0$ and $\mathrm{E}(\epsilon^2|X) = \sigma_\epsilon^2$, where $||\beta_0||_0 = \sum_{k=1}^d \mathbb{1}_{\{\beta_{0k}\neq 0\}} = s_\beta < n \ll d$.

(b) The missingness indicator R is conditionally independent of Y given X (*i.e.*, missing at random; MAR).

Main Takeaway:

- **Semi-parametrically efficient debiased estimator of $m_0(x) = x^T\beta_0$ with high-dimensional covariates, MAR outcomes, and heavy-tailed noises.**
- **Computationally efficient procedures with both Python (`Debias-Infer`) and R (`DebiasInfer`) implementations.**
- **Clear motivation for the proposed debiasing program by the rationale of bias-variance trade-offs.**
- **Comparative simulations and real-world applications with finite-sample performance guarantees.**

Motivation: Bias-Variance Trade-off

The conditional mean squared error can be decomposed as:

$$\mathrm{E}\left[\left(\sqrt{n}\, m^{\text{debias}}(x;\boldsymbol{w}) - \sqrt{n}\, m_0(x)\right)^2 \middle| \boldsymbol{X}\right]$$

$$= \underbrace{\sigma_\epsilon^2 \sum_{i=1}^n w_i^2 \pi(X_i)}_{\text{Main variance}} + \underbrace{\left[\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n w_i \pi(X_i) X_i - x\right)^T \sqrt{n}\left(\beta_0 - \widehat{\beta}\right)\right]^2}_{\text{Conditional bias}}$$

$$\underbrace{+\left(\beta_0 - \widehat{\beta}\right)^T \left[\sum_{i=1}^n w_i^2 \pi(X_i)\left(1-\pi(X_i)\right) X_i X_i^T\right]\left(\beta_0 - \widehat{\beta}\right).}_{\text{Asymptotic negligible variance}}$$

Idea: Design a convex program that solves for the debiasing weights $w_i, i = 1, ..., n$ in order to

- Minimize the estimated "**Main variance**" $\sum_{i=1}^n w_i^2 \widehat{\pi}_i$.
- Constrain the estimated upper bound of "**Conditional bias**"

$$\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^n w_i \widehat{\pi}_i X_i - x\right\|_\infty \sqrt{n}\left\|\beta_0 - \widehat{\beta}\right\|_1.$$

Methodology and Theory

To conduct statistical inference on $m_0(x) = x^T\beta_0$, we propose a debiasing method with the following procedures:

Step 1: Compute the **Lasso pilot estimate** $\widehat{\beta}$ with complete-case data

$$\widehat{\beta} = \arg\min_{\beta \in \mathbb{R}^d}\left[\frac{1}{n}\sum_{i=1}^n R_i (Y_i - X_i^T\beta)^2 + \lambda ||\beta||_1\right],$$

where $\lambda > 0$ is a regularization parameter.

Step 2: Obtain **consistent propensity score estimates** $\widehat{\pi}_i = \widehat{\mathrm{P}}(R_i = 1|X_i)$ for $i = 1, ..., n$ by any machine learning method (not necessarily a parametric model) on the data $\{(X_i, R_i)\}_{i=1}^n$.

Step 3: Solve for the debiasing weight vector $\widehat{\boldsymbol{w}} \equiv \widehat{\boldsymbol{w}}(x) = (\widehat{w}_1(x), ..., \widehat{w}_n(x))^T \in \mathbb{R}^n$ through a **debiasing program** defined as:

$$\min_{\boldsymbol{w}\in\mathbb{R}^n}\left\{\sum_{i=1}^n \widehat{\pi}_i w_i^2 : \left\|x - \frac{1}{\sqrt{n}}\sum_{i=1}^n w_i \widehat{\pi}_i X_i\right\|_\infty \leq \frac{\gamma}{n}\right\}, \tag{1}$$

where $\gamma > 0$ is a tuning parameter.

Step 4: Define the **debiased estimator** for $m_0(x)$ as:

$$\widehat{m}^{\text{debias}}(x;\widehat{\boldsymbol{w}}) = x^T\widehat{\beta} + \frac{1}{\sqrt{n}}\sum_{i=1}^n \widehat{w}_i(x) R_i\left(Y_i - X_i^T\widehat{\beta}\right). \tag{2}$$

Step 5: Construct the **asymptotic $(1-\tau)$-level confidence interval** as:

$$\left[\widehat{m}^{\text{debias}}(x;\widehat{\boldsymbol{w}}) \pm \Phi^{-1}\left(1-\frac{\tau}{2}\right)\cdot \sigma_\epsilon \cdot \sqrt{\frac{1}{n}\sum_{i=1}^n \widehat{\pi}_i \widehat{w}_i(x)^2}\right],$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of $\mathcal{N}(0,1)$.

Dual Formulation of the Debiasing Program (1):

$$\min_{\ell\in\mathbb{R}^d}\left\{\frac{1}{4n}\sum_{i=1}^n \widehat{\pi}_i\left[X_i^T\ell\right]^2 + x^T\ell + \frac{\gamma}{n}||\ell||_1\right\}. \tag{3}$$

- The relation between the primal solution $\widehat{\boldsymbol{w}}(x) \in \mathbb{R}^n$ and the dual solution $\widehat{\ell}(x) \in \mathbb{R}^d$ is $\widehat{w}_i(x) = -\frac{1}{2\sqrt{n}} \cdot X_i^T\widehat{\ell}(x)$ for $i = 1, ..., n$.
- The tuning parameter $\gamma > 0$ of the debiasing program (1) can be selected via cross-validations (CV) on the dual program (3).

Asymptotic Normality: $\sqrt{n}\left[\widehat{m}^{\text{debias}}(x;\widehat{\boldsymbol{w}}) - m_0(x)\right] \xrightarrow{d} \mathcal{N}\left(0, \sigma_m^2(x)\right)$, where $\sigma_{m,d}^2(x) := \sigma_\epsilon^2 x^T\left[\mathrm{E}\left(RXX^T\right)\right]^{-1} x$ and $\sigma_m^2(x) = \lim_{n\to\infty}\sigma_{m,d}^2(x)$. Here, $\sigma_{m,d}^2(x)$ attains the **semi-parametric efficiency bound** among all asymptotically linear estimators with MAR outcomes for any fixed d.

Inference on the Linear CATE

With $\mathbb{Y} = T\cdot Y(1) + (1-T)\cdot Y(0)$, the debiased estimator $\widehat{m}^{\text{debias}}(x;\widehat{\boldsymbol{w}}_{(1)}, \widehat{\boldsymbol{w}}_{(0)})$ becomes

$$x^T\left(\widehat{\beta}_{(1)} - \widehat{\beta}_{(0)}\right) + \frac{1}{\sqrt{n}}\sum_{i=1}^n\left[\widehat{w}_{i(1)} T_i\left(\mathbb{Y}_i - X_i^T\widehat{\beta}_{(1)}\right) - \widehat{w}_{i(0)}\left(1-T_i\right)\left(\mathbb{Y}_i - X_i^T\widehat{\beta}_{(0)}\right)\right],$$

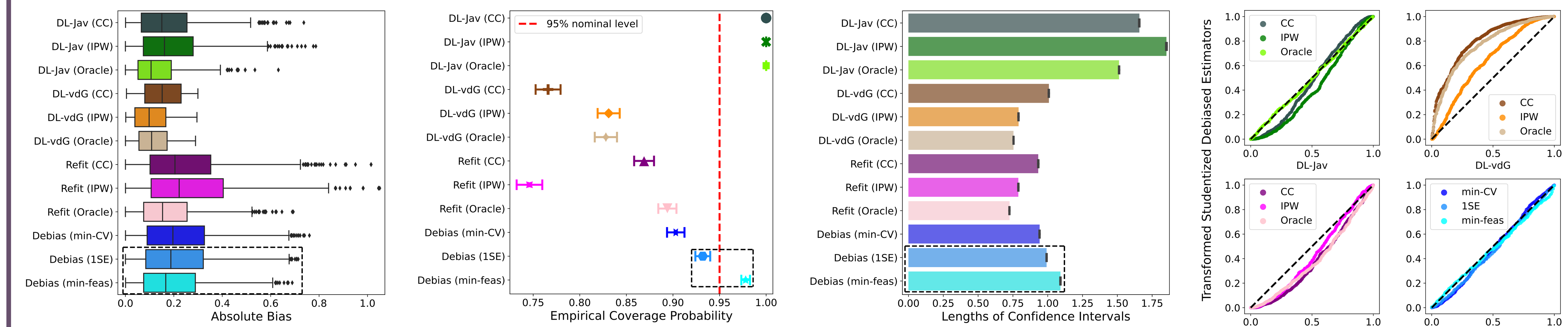
where the weight vectors $\widehat{\boldsymbol{w}}_{(1)}, \widehat{\boldsymbol{w}}_{(2)} \in \mathbb{R}^n$ come from

$$\arg\min_{\boldsymbol{w}_{(0)},\boldsymbol{w}_{(1)}\in\mathbb{R}^n}\sum_{i=1}^n\left[\widehat{\pi}_i w_{i(1)}^2 + (1-\widehat{\pi}_i) w_{i(0)}^2\right]$$

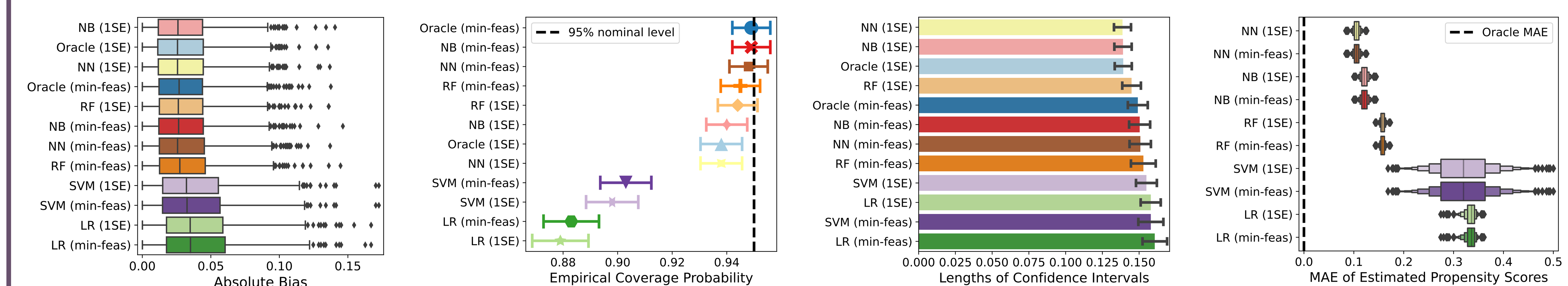
$$\text{s.t.}\ \left\|x - \frac{1}{\sqrt{n}}\sum_{i=1}^n w_{i(1)}\cdot\widehat{\pi}_i\cdot X_i\right\|_\infty \leq \frac{\gamma_1}{n}\ \text{and}\ \left\|x - \frac{1}{\sqrt{n}}\sum_{i=1}^n w_{i(0)}\left(1-\widehat{\pi}_i\right) X_i\right\|_\infty \leq \frac{\gamma_2}{n}.$$

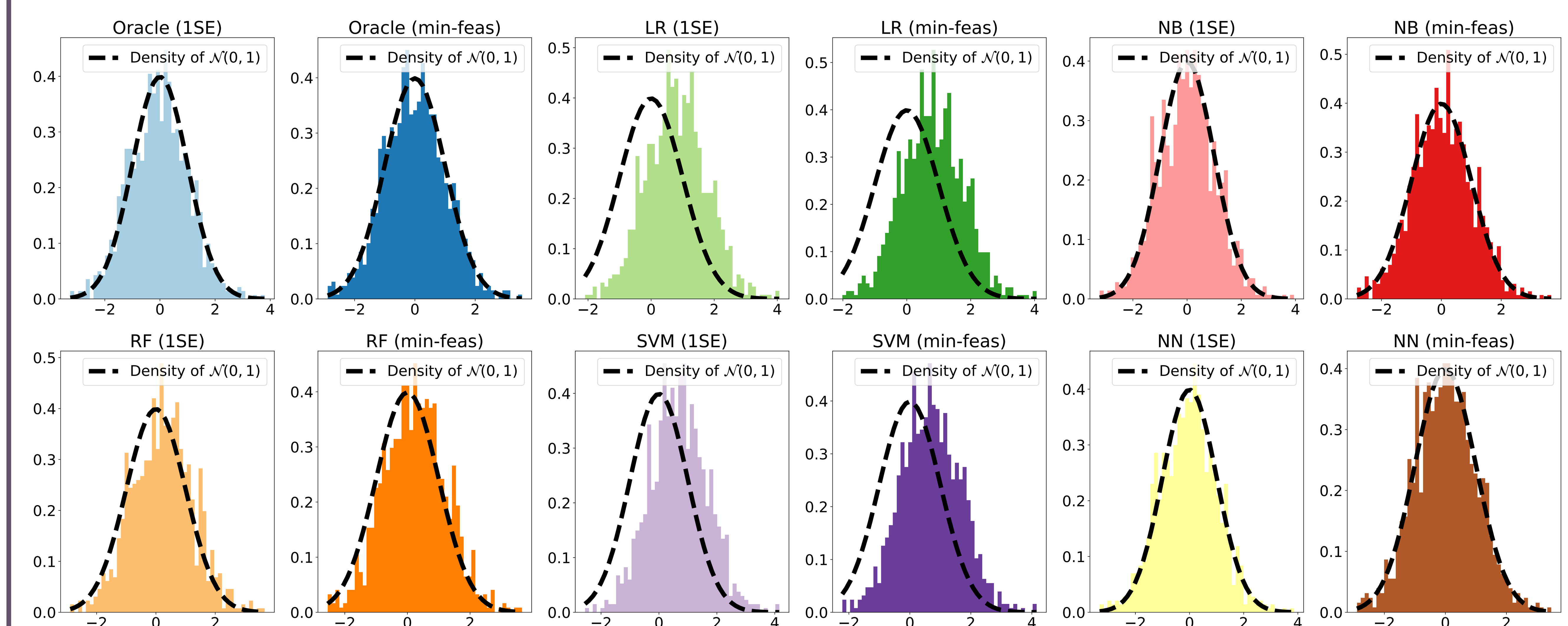
Simulation Studies and Real-World Applications

Comparisons With Existing Methods: DL-Jav, DL-vdG, and Refit are run on complete-case (CC), inverse probability weighting (IPW), and oracle data. Our proposed methods (Debias) under three CV-criteria are only run on the data with missing outcomes.



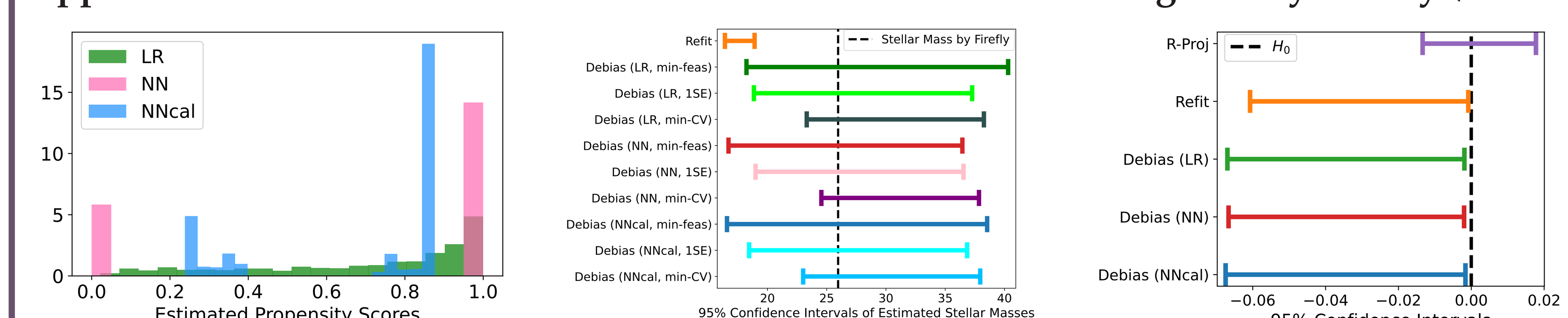
Proposed Debiasing Method With Nonparametrically Estimated Propensity Scores: LR (Lasso-type Logistic regression), NB (Gaussian Naive Bayes), RF (Random Forests), SVM (Support Vector Machine with Gaussian Radial Bases), and NN (Neural Networks).





$\sqrt{n}\left[\widehat{m}^{\text{debias}}(x) - m_0(x)\right]/\widehat{\sigma}_m(x)$

Applications to Stellar Mass Inference of Galaxies in the Sloan Digital Sky Survey (SDSS-IV, DR16):



Middle Panel: Inference on the stellar mass of a newly observed galaxy based on its spectroscopic and photometric properties.
Right Panel: Testing the negative correlation between the stellar mass and the distance to nearby cosmic filament structures.