Robust Optimization and Inference on Manifolds

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Presented by *Yikun Zhang*
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Background: Mean Estimation and Robust Statistics

Problem: Given a random sample $\{X_1, ..., X_n\} \sim P$, consider estimating the population mean $\mu = \mathbb{E}_P(X_i) = \int x \, dP$.

- We want to construct an estimator $\widehat{\mu}_n \equiv \widehat{\mu}_n(X_1, ..., X_n)$.

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Most popular estimator: the sample mean $\bar{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} X_i = \int x \, dP_n$,

where P_n is the empirical distribution.

Consistency: By the (strong) law of large number,

$$\lim_{n \to \infty} \bar{\mu}_n = \mu \quad \text{with probability one.}$$

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Drawbacks:

- Require strict assumptions on P for tight confidence bounds.
- Sensitive to outliers.
- ...

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Given a confidence level $\delta \in (0,1)$, what is the smallest $\epsilon \equiv \epsilon(n, \delta)$ such that

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Consider the sample mean $\bar{\mu}_n$:

- If we know $\sigma^2 = \mathbb{E}_P(X_i - \mu)^2 < \infty$, then by Chebyshev's inequality,

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- If P is sub-Gaussian, i.e., $\mathbb{E}_P \exp\left[\lambda(X - \mu)\right] \leq \exp\left(\frac{\sigma^2 \lambda^2}{2}\right)$, then

$$|\bar{\mu}_n - \mu| \leq \sigma \sqrt{\frac{2\log(2/\delta)}{n}} \quad \text{with probability at least } 1 - \delta. \tag{2}$$

Theorem (Theorem 1 in Lugosi and Mendelson 2019a)

Let $n > 5$ be an integer, $\sigma > 0$, and $\delta \in \left(\frac{e^{-n}}{2}, \frac{1}{2} \right)$. Then, for any mean estimator $\widehat{\mu}_n$, there exists a distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$ such that

$$\mathbb{P}\left\{ |\widehat{\mu}_n - \mu| > \sigma\sqrt{\frac{\log\left(1/(2\delta)\right)}{4n}} \right\} \geq \delta.$$

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Without sub-Gaussianity (i.e., P is heavy-tailed), the $\sqrt{1/\delta}$-bound is the best that $\bar{\mu}_n$ can achieve:

- for any $\delta \in (0, 1)$, there is a distribution with variance σ^2 such that

$$\mathbb{P}\left(|\bar{\mu}_n - \mu| > \sigma\sqrt{\frac{C}{n\delta}} \right) > \delta \quad \text{for some constant } C > 0.$$

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Question: Is there any estimator $\widehat{\mu}_n$ that can achieve the (sub-gaussian) $\sqrt{\log(1/\delta)}$-bound (2) for all distributions with finite variance?

Median-of-Means (Nemirovskij and Yudin, 1983; Jerrum et al., 1986; Alon et al., 1999): Divide the random sample $\{X_1, ..., X_n\}$ into m groups $B_1, ..., B_m$ with (roughly) equal size $B = \lfloor \frac{n}{m} \rfloor$ and define

$$\widehat{\mu}_{MM} \equiv \texttt{Median}\left(Z_1, ..., Z_m\right), \tag{3}$$

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- The MoM estimator is consistent as long as $B \to \infty$ as $n \to \infty$.
- For any $\delta \in (0, 1)$, if $m = \lfloor 8 \log(1/\delta) \rfloor$, then

$$|\widehat{\mu}_{MM} - \mu| \le \sigma \sqrt{\frac{32 \log(1/\delta)}{n}} \quad \text{with probability at least } 1 - \delta.$$

See Theorem 2 in Lugosi and Mendelson (2019a) and Proposition 1 in Yen-Chi's notes (http://faculty.washington.edu/yenchic/short_note/note_MoM.pdf).

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- However, if $\tau = \mathbb{E}_P\left[(X_i - \mu)^3\right] < \infty$ exists, we may take $m = \frac{2\sigma^3}{\tau}\sqrt{n}$ to achieve the sub-Gaussian performance; see Theorem 4 in Lugosi and Mendelson (2019a).

Discussions on the MoM Estimator

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  - Other mean estimators that attain the sub-Gaussian bound include

    - Catoni's estimator (Catoni, 2012): the solution to $\sum\limits_{i=1}^{n} \Psi\left(\alpha(X_i - y)\right) = 0$, where $\Psi : \mathbb{R} \to \mathbb{R}$ is an increasing odd function.

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    - Trimmed mean (Tukey and McLaughlin, 1963): $\widehat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} \phi_{\alpha,\beta}(X_i)$ with
      $$\phi_{\alpha,\beta}(x) = \begin{cases} \alpha & \text{if } x < \alpha, \\ x & \text{if } \alpha \leq x \leq \beta, \\ \beta & \text{if } x > \beta. \end{cases}$$

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- MoM can be used even if P only has a finite moment $\mathbb{E}_P\left[|X_i - \mu|^{1+\gamma}\right]$ of order $1 + \gamma$ with $\gamma \in (0, 1)$ (Bubeck et al., 2013; Devroye et al., 2016).

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- Consider a set $\mathcal{D}_s = \{\mathcal{Y} = \{Y_1, ..., Y_n\} : |\boldsymbol{Y}| = n, |\boldsymbol{X} \cap \boldsymbol{Y}| = n - s\}$. The robustness of $\widehat{\mu}_n(\mathcal{X})$ with $\mathcal{X} = \{X_1, ..., X_n\}$ can be measured by the *breakdown point* as (Huber, 2004):

$$\epsilon^* \left(\widehat{\mu}_n(\mathcal{X})\right) = \max \left\{ \frac{s}{n} : ||\widehat{\mu}_n(\mathcal{Y})|| < \infty \text{ for all } \mathcal{Y} \in \mathcal{X}_s \right\}.$$

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- The MoM estimator $\widehat{\mu}_{MM}(\mathbf{X})$ has the breakdown point as $\frac{m-1}{2n}$, where m is the number of blocks (Rodriguez and Valdora, 2019).

Let $\{X_1, ..., X_n\}$ be an i.i.d. sample in $\mathbb{R}^d$ with $\boldsymbol{\mu} = \mathbb{E}(X_i)$ and $\Sigma = \mathbb{E}\left[(X_i - \boldsymbol{\mu})(X_i - \boldsymbol{\mu})^T\right]$.

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Definition

We say that a mean estimator $\widehat{\boldsymbol{\mu}}_n$ is *sub-Gaussian* if, for $\delta \in (0, 1)$,

$$||\widehat{\boldsymbol{\mu}}_n - \boldsymbol{\mu}|| \leq C \left(\sqrt{\frac{\text{Tr}(\Sigma)}{n}} + \sqrt{\frac{2\lambda_{\max} \log(1/\delta)}{n}} \right) \text{ with probability at least } 1 - \delta$$

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Question: Can the (multivariate) MoM estimator attain the above sub-Gaussian bound?

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- *Coordinate-wise median*: for any $\delta \in (0, 1)$, take $m = \lfloor 8 \log(1/\delta) \rfloor$,

$$||\widehat{\boldsymbol{\mu}}_n - \boldsymbol{\mu}|| \leq \sqrt{\frac{32 \text{Tr}(\Sigma) \log(d/\delta)}{n}} \quad \text{with probability at least } 1 - \delta.$$

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Multivariate Median-of-Means

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- The estimators that truly yield the sub-Gaussian performance are
  1. *Catoni-Giulini estimator* (Catoni and Giulini, 2018):
     $\widehat{\boldsymbol{\mu}}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \cdot \min\left(1, \frac{1}{\alpha||X_i||}\right)$ with tuning parameter $\alpha > 0$.
  2. the *median-of-means tournaments* (Lugosi and Mendelson, 2019b).

Geometric Median on Manifolds

For a metric space $(\mathcal{M}, \rho)$, the *geometric median p^** of $p_1, ..., p_m \in \mathcal{M}$ minimizes the sum of distances to the points (Minsker, 2015):

$$p^* = \text{med}(p_1, ..., p_m) = \operatorname*{arg\,min}_{p \in \mathcal{M}} \frac{1}{m} \sum_{k=1}^{m} \rho(p, p_k). \tag{4}$$

assuming that p^* exists. It is unique (Theorem 1 in Fletcher et al. 2008)
(i) if the sectional curvatures of $\mathcal{M}$ is nonpositive or
(ii) if the section curvatures of $\mathcal{M}$ are bounded by $\Delta > 0$ and
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When $\mathcal{M}$ is a manifold, there are two different ways to define ρ.

❶ (*Extrinsic distance*) Given an embedding $J : \mathcal{M} \to \mathbb{R}^d$ into the ambient space $\mathbb{R}^d$,

$$\rho(p, q) = ||J(p) - J(q)|| \quad \text{with } ||\cdot|| \text{ being the Euclidean norm in } \mathbb{R}^d.$$

❷ (*Intrinsic distance*) Take ρ as the geodesic distance arising from a Riemannian structure on $\mathcal{M}$.

To compute $p^* = \arg\min\limits_{p \in \mathcal{M}} \sum\limits_{k=1}^{m} \rho(p, p_k) \equiv h(p)$, we leverage the Ostresh's modification of the Weiszfeld Algorithm (Weiszfeld, 1937; Ostresh Jr, 1978; Fletcher et al., 2008):

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1. Compute the (Riemannian) gradient

$$\nabla h(p) = -\sum_{k=1}^{m} \frac{\text{Log}_p(p_k)}{\rho(p, p_k)} \quad \text{when } p \neq p_k.$$

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2. Apply the gradient descent iteration

$$p^{(t+1)} \leftarrow \text{Exp}_{p^{(t)}}\left(\eta' \cdot v^{(t)}\right) \text{ with } v^{(t)} = \sum_{k \in I_t} \frac{\text{Log}_{p^{(t)}}(p_k)}{\rho(p^{(t)}, p_k)} \cdot \left(\sum_{k \in I_t} \frac{1}{\rho(p^{(t)}, p_k)}\right)^{-1}$$

where $\eta' \in [0, 2]$ is the step size and $I_t = \left\{k \in \{1, ..., m\} : p_k \neq p^{(t)}\right\}$.

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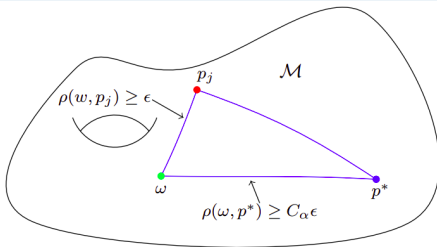**Convergence**: $\lim\limits_{t \to \infty} p^{(t)} = p^*$ when $\mathcal{M}$ has a nonnegative sectional curvature.

Lemma (Lemma 2.1 in Minsker 2015; Lin et al. 2020)

Let $p_1, ..., p_m \in \mathcal{M}$ and $p^* = med(p_1, ..., p_m)$ as in (4).
(a) Let ρ be the extrinsic distance of an embedding $J : \mathcal{M} \to \widetilde{\mathcal{M}} \subset \mathbb{R}^d$, $w \in \mathcal{M}$, ψ be the angle between $J(w) - J(p^*)$ and the tangent space $T_{J(p^*)}\widetilde{\mathcal{M}}$, and

$$C_\alpha = \frac{1 - \alpha}{\sqrt{1 - 2\alpha}\cos\psi - \alpha\sin\psi} \quad with \quad \alpha \in \left(0, \cot\psi\tan\frac{\psi}{2}\right).$$

If $\rho(w, p^*) \geq C_\alpha\epsilon$, then there exists an index set $T \subset \{1, ..., m\}$ with $|T| \geq \alpha m$ such that $\rho(p_j, w) \geq \epsilon$ for any $j \in T$.
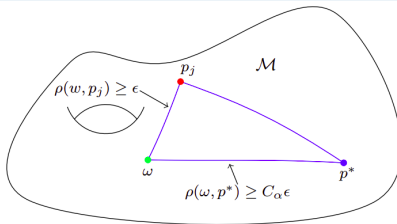
Lemma (Continued)

Let $p_1, ..., p_m \in \mathcal{M}$ and $p^* = med(p_1, ..., p_m)$ as in (4).
(b) Let ρ be an intrinsic distance on $\mathcal{M}$ with respect to some Riemannian structure, $w \in \mathcal{M}$, the logarithm map Log_{p^*} be K-Lipschitz continuous from $B(w, \epsilon)$ to $T_{p^*}\mathcal{M}$, and

$$C_\alpha = K(1 - \alpha)\sqrt{\frac{1}{1 - 2\alpha}} \quad with \quad \alpha \in \left(0, \frac{1}{2}\right).$$

If $\rho(w, p^*) \geq C_\alpha \epsilon$, then there exists an index set $T \subset \{1, ..., m\}$ with $|T| \geq \alpha m$ such that $\rho(p_j, w) \geq \epsilon$ for any $j \in T$.

Proof. Let $L(p) = \sum\limits_{k=1}^{m} \rho(p, p_k)$. Consider the geodesic curve $\gamma(t) = \text{Exp}_{p^*}(tv)$ with $v = \text{Log}_{p^*} w \in T_{p^*}\mathcal{M}$. Then,

$$dL_{p^*}(v) = \lim_{t \to 0^+} \frac{L(\gamma(t)) - L(\gamma(0))}{t} = \lim_{t \to 0^+} \frac{L(\gamma(t)) - L(p^*)}{t} \geq 0,$$

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$$\frac{dL_{p^*}(v)}{||v||} = -\sum_{j: p_j \neq p^*} \frac{\langle v, v_j \rangle}{||v|| \, ||v_j||} + \sum_{j=1}^{m} \mathbb{1}_{\{p_j = p^*\}},$$

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where $v_j = \text{Log}_{p^*} p_j$. Assume, by contradiction and without the loss of generality, that

$$\rho(w, p_j) \leq \epsilon \quad \text{for } j = 1, ..., \lfloor (1-\alpha)m \rfloor + 1,$$
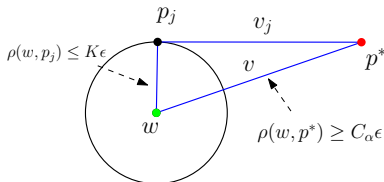
while $\rho(w, p^*) \geq C_\alpha \epsilon$.

By the Lipschitz continuity of Log_{p^*} from $B(w, \epsilon)$ to $T_{p^*}\mathcal{M}$, for $j = 1, ..., \lfloor (1 - \alpha)m \rfloor + 1$,

$$\left| \left| v_j - v \right| \right| = \left| \left| \text{Log}_{p^*} p_j - \text{Log}_{p^*} w \right| \right| \leq K \cdot d_g(p_j, w) = K \cdot \rho(p_j, w) \leq K\epsilon.$$

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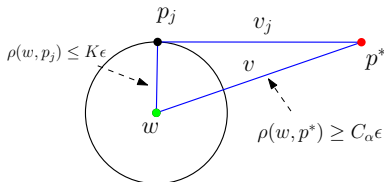
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This implies that $\sin(\widehat{v_j, v}) \leq \frac{K}{C_\alpha}$.

Thus, whenever $C_\alpha > K(1-\alpha)\sqrt{\frac{1}{1-2\alpha}}$, we have that

$$\frac{dL_{p^*}(v)}{||v||} = -\sum_{j:p_j \neq p^*} \cos(\widehat{v_j, v}) + \sum_{j=1}^{m} \mathbb{1}_{\{p_j = p^*\}} \leq -(1-\alpha)m\sqrt{1 - \frac{K^2}{C_\alpha^2}} + \alpha m < 0,$$

which is a contradiction. $\qquad\qquad\square$

There are many Riemannian manifolds with *K*-Lipschitz continuous logarithm map.

1. *d-dimensional sphere* $S^d = \left\{ p \in \mathbb{R}^{d+1} : ||p|| = 1 \right\}$: $\mathrm{Log}_p(\cdot)$ on S^d is 2-Lipschitz continuous from $B(p, \pi/2)$ to $T_p S^d$ for all $p \in S^d$.

2. *Planar shape space* $\Sigma_2^k = S^{2k-3}/S^1$: $\mathrm{Log}_p(\cdot)$ on Σ_2^k is 2-Lipschitz continuous from $B(p, \pi/4)$ to $T_p \Sigma_2^k$ for all $p \in S^d$.

3. *Positive definite matrices* $PD(n) \subset \mathbb{R}^{n \times n}$: $\mathrm{Log}_p(\cdot)$ is 1-Lipschitz continuous at any $p \in PD(n)$.

Robust Optimization on Manifolds

Let Q be a probability distribution on some space $\mathcal{X}$ and $\mathcal{M}$ be a manifold. Consider estimating the *population parameter*

$$\mu = \arg\min_{p \in \mathcal{M}} L^*(p),$$

where, for some loss function L,

$$L^*(p) = \int_{\mathcal{X}} L(p, x) \, Q(dx).$$

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- *Fréchet mean*: $\arg\min_{p \in \mathcal{M}} \int_{\mathcal{M}} \rho^2(p, x)\, Q(dx)$ with Q supported on $\mathcal{M}$.
- *Geometric median*: $\arg\min_{p \in \mathcal{M}} \int_{\mathcal{M}} \rho(p, x)\, Q(dx)$ with Q supported on $\mathcal{M}$.

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In practice, given a random sample $\{X_1, ..., X_n\} \sim Q$, the population parameter μ can be estimated by the *empirical risk estimator*

$$\widehat{\mu}_n = \arg\min_{p \in \mathcal{M}} \frac{1}{n} \sum_{i=1}^{n} L(p, X_i).$$

Objective: $\mu = \underset{p \in \mathcal{M}}{\arg\min} \int_{\mathcal{X}} L(p, x) \, Q(dx)$.

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③ The final estimator is $\widehat{\mu}^* = \underset{p \in \mathcal{M}}{\arg\min} \sum_{j=1}^{m} \rho(p, \mu_j)$.

$\widehat{\mu}^*$ inherits the desired robustness properties in estimating the population parameter μ.

Theorem (Theorem 3.1 in Lin et al. 2020)

Let $\mu_1, ..., \mu_m$ be some independent estimators of μ and $\mu^* = med(\mu_1, ..., \mu_m)$.

(a) If $\rho(p, q) = ||J(p) - J(q)||$ with $J : \mathcal{M} \to \widetilde{\mathcal{M}} \subset \mathbb{R}^d$, we assume that for any $w \in \mathcal{M}$, the angle between $J(w) - J(\mu^*)$ and the tangent space $T_{J(\mu^*)}\widetilde{\mathcal{M}}$ is no bigger than $\bar{\psi}$. For any $\alpha \in \left(0, \cot \bar{\psi} \tan \frac{\bar{\psi}}{2}\right)$, set $\bar{C}_\alpha = \frac{1-\alpha}{\sqrt{1-2\alpha}\cos\bar{\psi} - \alpha\sin\bar{\psi}}$.

(b) Let ρ be an intrinsic distance on $\mathcal{M}$. Assume that $\mathtt{Log}_{\mu^*}$ is K-Lipschitz continuous from $B(\mu^*, \epsilon)$ to $T_{\mu^*}\mathcal{M}$. For any $\alpha \in \left(0, \frac{1}{2}\right)$, set

$$\bar{C}_\alpha = K(1-\alpha)\sqrt{\frac{1}{1-2\alpha}}.$$

Under (a) or (b), if $\mathbb{P}\left\{\rho(\mu_j, \mu) > \epsilon\right\} \leq \eta$ for $j = 1, ..., m$ with $\eta < \alpha$, then

$$\mathbb{P}\left\{\rho(\mu^*, \mu) > \bar{C}_\alpha\epsilon\right\} \leq \exp\left[-m \cdot \phi(\alpha, \eta)\right],$$

where $\phi(\alpha, \eta) = (1-\alpha)\log\left(\frac{1-\alpha}{1-\eta}\right) + \alpha\log\frac{\alpha}{\eta}$.

Proof. Note that when $\psi < \bar{\psi}$, we have that $C_\alpha \leq \bar{C}_\alpha$ and $\cot \bar{\psi} \tan \frac{\bar{\psi}}{2} \leq \cot \psi \tan \frac{\psi}{2}$. By the previous lemma,

$$
\mathbb{P}\left\{\rho(\mu^*, \mu) > \bar{C}_\alpha \epsilon\right\} \leq \mathbb{P}\left\{\rho(\mu^*, \mu) > C_\alpha \epsilon\right\}
$$

$$
\leq \mathbb{P}\left(\sum_{j=1}^{m} \mathbb{1}_{\{\rho(\mu_j, \mu) > \epsilon\}} > \alpha m\right)
$$

$$
\leq \exp\left[-m \cdot \phi(\alpha, \eta)\right],
$$

where we leverage a coupling result (Lemma 23 in Lerasle and Oliveira 2011) and Chernoff's bound to obtain the last inequality. $\square$

Recall our mean estimation problem via the MoM estimator.

1. Partition the dataset $\{X_1, ..., X_n\} \subset \mathbb{R}^d$ into m groups $U_1, ..., U_m$ and compute the within-group means $Z_i = \frac{1}{|U_i|} \sum_{j \in U_i} X_j$.

2. Define the geometric median estimator $\widehat{\boldsymbol{\mu}}_n = \arg\min_{\boldsymbol{p} \in \mathbb{R}^d} \frac{1}{m} \sum_{j=1}^{m} \left|\left| \boldsymbol{p} - \boldsymbol{Z}_j \right|\right|$.

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Set $\alpha_* = \frac{7}{18}$ and $\eta_* = 0.1$. For any $\delta \in (0, 1)$, we take

$$m = \left\lfloor \frac{\log(1/\delta)}{\phi(\alpha_*, \eta_*)} \right\rfloor + 1 \leq \lfloor 3.5 \log(1/\delta) \rfloor + 1.$$

Then,

$$||\widehat{\boldsymbol{\mu}}_n - \boldsymbol{\mu}|| \leq 11 \sqrt{\frac{\text{Tr}(\Sigma) \log(1.4/\delta)}{n}} \quad \text{with probability at least } 1 - \delta.$$

See Corollary 4.1 in Minsker (2015).

The geometric median of subset optimizers is $\widehat{\mu}^* = \underset{p \in \mathcal{M}}{\arg\min} \sum_{j=1}^{m} \rho(p, \mu_j)$.

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- However, the within-group sample size $\lfloor n/m \rfloor$ should also be large so that each subset estimator behaves well, i.e.,

$$\mathbb{P}\left\{ \rho(\mu_j, \mu) > \epsilon \right\} \leq \eta \quad \text{for } j = 1, ..., m \text{ with a small } \eta.$$

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For a given confidence level $\delta \in (0, 1)$, one can determine the number of subsets, m, to achieve a small η.

- However, in practice, η may depend on the unknown parameter; see Example 2 in Lin et al. (2020).

Simulations and Real-World Applications

Problem: Estimate the intrinsic and extrinsic means of the von Mises Fisher distribution in the presence of outliers.

$$\text{vMF}(\mu, \kappa) \text{ on } \mathbb{R}^d \quad \sim \quad f_d(x; \mu, \kappa) = \frac{\kappa^{d/2-1}}{(2\pi)^{d/2}\mathcal{I}_{d/2-1}(\kappa)} \cdot \exp(\kappa\mu^T x).$$

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Computing sample statistics on $\{p_1, ..., p_n\} \subset S^d$.

• *Intrinsic mean*: $\arg\min\limits_{x \in S^d} \sum\limits_{i=1}^{n} \arccos^2\left(x^T p_i\right)$.

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- *Intrinsic mean*: $\arg\min_{x \in S^d} \sum_{i=1}^{n} \arccos^2(x^T p_i)$. By the Lagrangian multiplier, the intrinsic mean on S^d can be obtained by a fixed-point iteration

$$\mu^{(t+1)} \leftarrow \frac{\sum_{i=1}^{n} \gamma_i(\mu^{(t)}) p_i}{||\sum_{i=1}^{n} \gamma_i(\mu^{(t)}) p_i||} \quad \text{for } t = 0, 1, ... \text{ with } \gamma_i(x) = \frac{\arccos(x^T p_i)}{\sqrt{1 - (x^T p_i)^2}}.$$

Notes: Its derivation is similar to our directional mean shift algorithm; see Section 2.2 in Zhang and Chen (2021).

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Notes: Its derivation is similar to our directional mean shift algorithm; see Section 2.2 in Zhang and Chen (2021).

• *Intrinsic median*: $\arg\min\limits_{x \in S^d} \sum\limits_{i=1}^{n} \arccos\left(x^T p_i\right)$ by the modified Weiszfeld's algorithm.

• *Extrinsic mean*: $\mathcal{P}\left(\frac{1}{n}\sum_{i=1}^{n} J(p_i)\right)$, where $J : \mathcal{M} \to \widetilde{\mathcal{M}} \subset \mathbb{R}^d$ is the embedding map and $\mathcal{P} : \mathbb{R}^d \to \mathcal{M}$ is the projection map. When $\mathcal{M} = S^d$, the extrinsic mean is $\frac{\sum_{i=1}^{n} p_i}{||\sum_{i=1}^{n} p_i||}$, i.e., the spherical mean.

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• *Extrinsic median*: $\arg\min_{p \in S^d} \sum_{i=1}^{n} ||x - p_i||$ by the projected gradient descent on S^d (Weiszfeld's algorithm).

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Evaluation metric. Repeat the simulation for several times and compute the averages based on the following measures:

- the intrinsic distance $\rho(\mu^*, \mu)$ from the true mean μ to the geometric median of subset means μ^*.

- the average intrinsic distance $\overline{\rho(\mu_i, \mu)} = \frac{1}{m}\sum\limits_{k=1}^{m} \rho(\mu_i, \mu)$ from μ to the subset means $\mu_k, k = 1, ..., m$.

k	$\overline{\rho(\hat{\mu}, \mu)}$	$\overline{\rho(\mu^*, \mu)}$	$\overline{\rho(\mu_i, \mu)}$	$\overline{\rho(\mu^*, \mu)}$	$\overline{\rho(\mu_i, \mu)}$
0	0.0597	0.0583	0.0947	0.0514	0.1496
5	0.0647	0.0615	0.1159	0.0531	0.1652
10	0.1194	0.1116	0.1414	0.1018	0.2113
15	0.1819	0.1731	0.1973	0.1631	0.2419
	sample mean (m=1)	m=5		m=15	

k	$\overline{\rho(\mu^*, \mu)}$	$\overline{\rho(\mu_i, \mu)}$	$\overline{\rho(\hat{m}, \mu)}$	$\overline{\rho(\mu_i, \mu)}$
0	0.0455	0.2118	0.0424	0.2829
5	0.0453	0.2350	0.0447	0.2959
10	0.0776	0.2501	0.0614	0.3259
15	0.1383	0.2954	0.0925	0.3738
	m=30		sample median (m=60)	

Figure 1: Estimating the mean of vMF($\mu, \kappa = 30$) on S^2 with k being the number of outliers and ρ being the intrinsic distance.

k	$\overline{\rho(\hat{\mu}, \mu)}$	$\overline{\rho(\mu^*, \mu)}$	$\overline{\rho(\mu_i, \mu)}$	$\overline{\rho(\mu^*, \mu)}$	$\overline{\rho(\mu_i, \mu)}$
0	0.0396	0.0399	0.1186	0.0384	0.2570
10	0.0565	0.0541	0.1258	0.0514	0.2669
20	0.0897	0.0900	0.1462	0.0834	0.2827
40	0.1656	0.1678	0.2082	0.1596	0.3376
	Sample mean (m=1)	m=10		m=50	

k	$\overline{\rho(\mu^*, \mu)}$	$\overline{\rho(\mu_i, \mu)}$	$\overline{\rho(\hat{m}, \mu)}$	$\overline{\rho(\mu_i, \mu)}$
0	0.0398	0.3590	0.0387	0.4896
10	0.0469	0.3676	0.0457	0.4978
20	0.0760	0.3896	0.0682	0.5301
40	0.1513	0.5176	0.1305	0.5987
	m=100		sample median (m=200)	

Figure 2: Estimating the mean of vMF($\mu, \kappa = 30$) on S^7 with k being the number of outliers and ρ being the intrinsic distance.

Principal Geodesic Analysis (Fletcher and Joshi, 2007; Lazar and Lin, 2017):

1. Compute the center of the data.
2. Successively find some orthogonal tangent vectors at the center so that their exponentiated space best fits the data according to the intrinsic sum of squared residuals.
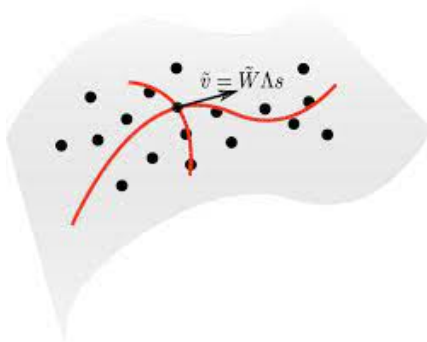
Robust Principal Geodesic Analysis (RPGA):

1. Divide the data $\{X_1, ..., X_n\}$ into m groups $U_1, ..., U_m$, compute the within-group intrinsic mean μ_j, and take $\mu^* = \text{med}(\mu_1, ..., \mu_m)$.

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2. Calculate $\boldsymbol{V}_k = \left\{ \text{vec}\left(\text{Log}_{\mu^*}(X_j) \right) : j \in U_i \right\}$ and the sample covariance matrix Σ_k of points in $\boldsymbol{V}_k$ for $k = 1, ..., m$.

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3. Compute
$$\widehat{\Sigma} = \text{med}\left(\Sigma_1, ..., \Sigma_m\right),$$
where the median is taken with respect to the Frobenius norm $||A||_F = \text{Tr}(A^T A)$.

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3. Compute
$$\widehat{\Sigma} = \text{med} \left(\Sigma_1, ..., \Sigma_m \right),$$
where the median is taken with respect to the Frobenius norm $||A||_F = \text{Tr}(A^T A)$.

4. Compute the eigenvectors of $\widehat{\Sigma}$, $\{\omega_1, ..., \omega_6\}$, arranged in order by largest to smallest eigenvalues.

k	PGA	RPGA	RPGA	RPGA
0	0.4206	0.4265	0.4259	0.4320
5	0.4529	0.4465	0.4314	0.4342
10	0.4541	0.4438	0.4508	0.4374
15	0.4540	0.4445	0.4492	0.4442
20	0.4527	0.4473	0.4507	0.4496
m groups	m=5	m=10	m=15

k	PGA	RPGA	RPGA	RPGA
0	0.2629	0.2686	0.2691	0.2751
5	0.2924	0.2870	0.2803	0.2795
10	0.2963	0.2838	0.2925	0.2791
15	0.2994	0.2835	0.2758	0.2850
20	0.3041	0.2841	0.2889	0.2775
m groups	m=5	m=10	m=15

k	PGA	RPGA	RPGA	RPGA
0	0.1472	0.1497	0.1533	0.1608
5	0.1919	0.1801	0.1600	0.1588
10	0.2242	0.2102	0.1940	0.1743
15	0.2208	0.2149	0.2134	0.2079
20	0.2305	0.2259	0.2169	0.2206
m groups	m=5	m=10	m=15

Figure 3: Average mean sum of square residuals to explanatory submanifolds computed with k outliers to data without outliers in $PD(3)$.
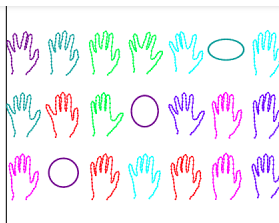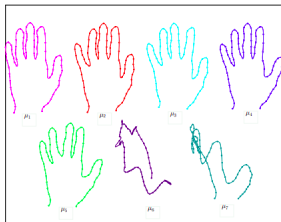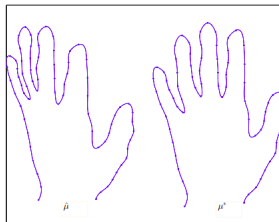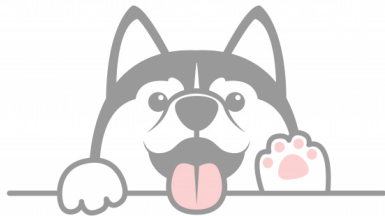
(A) Hand Shape Data with 3 outliers

(B) $m = 7$ subsets

(C) Subset means, μ_i

(D) Sample mean, $\hat{\mu}$ and geometric median, μ^*

Figure 4: Median-of-means on hand shape data.

Thank you!

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Median-of-Means Tournaments

Given an i.i.d. sample $\{X_1, ..., X_n\} \subset \mathbb{R}^d$, we partition it into m groups $U_1, ..., U_m$ and compute the within-group means $Z_i = \frac{1}{|U_i|} \sum_{j \in U_i} X_j$.

For each $a \in \mathbb{R}^d$, let

$$T_a = \left\{ x \in \mathbb{R}^d : \exists J \subset \{1, ..., m\} \text{ with } |J| \geq m/2 \text{ such that for all } j \in J, ||Z_j - x|| \leq ||Z_j - a|| \right\}$$

and define the "median-of-means tournaments" estimator by

$$\widehat{\mu}_n \in \operatorname*{arg\,min}_{a \in \mathbb{R}^d} \operatorname{radius}(T_a),$$

where $\operatorname{radius}(T_a) = \sup_{x \in T_a} ||x - a||$.

Theorem (Lugosi and Mendelson 2019b)

Let $\delta \in (0, 1)$ and $k = \lceil 200 \log(2/\delta) \rceil$. If $X_1, ..., X_n$ are i.i.d. random vectors in $\mathbb{R}^d$ with mean $\mu \in \mathbb{R}^d$ and covariance matrix Σ, then for all n,

$$||\widehat{\mu}_n - \mu|| \leq \max \left\{ 960 \sqrt{\frac{Tr(\Sigma)}{n}}, 240 \sqrt{\frac{\lambda_{\max} \log(2/\delta)}{n}} \right\}$$

with probability at least $(1 - \delta)$, where $\lambda_{\max}$ is the maximal eigenvalue of Σ.