Nonparametric Inference on Causal Effects of Continuous Treatments: With and Without the Positivity Condition

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Pinterest
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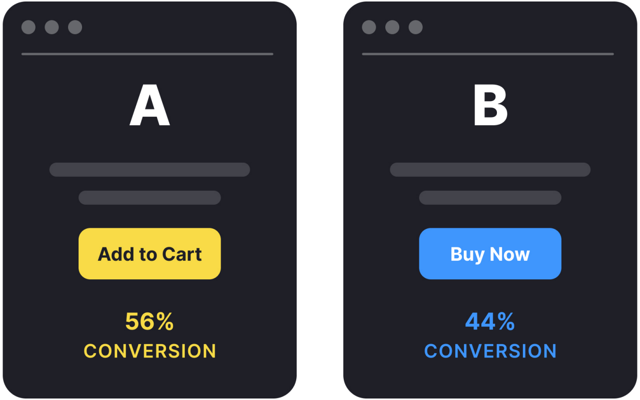
Fundamental Problem of Causal Inference

- Study the causal effect of a treatment $T \in \mathcal{T}$ on the outcome of interest $Y \in \mathcal{Y}$.



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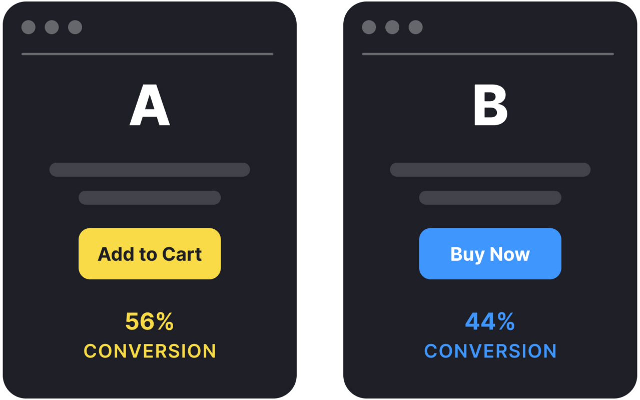
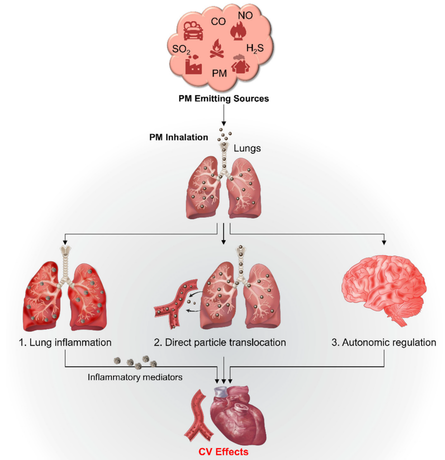


- The treatment variable T is *binary*, *i.e.*, $\mathcal{T} = \{0, 1\}$.
- Only one potential outcome, $Y(1)$ or $Y(0)$, can be observed for each individual.
- The common causal estimand is the average treatment effect $\mathbb{E}[Y(1)] - \mathbb{E}[Y(0)]$.

- We want to study the causal effects of $PM_{2.5}$ levels on Cardiovascular Mortality Rates (CMRs).



Biological pathways associated with particulate matter (PM) and cardiovascular disease (Miller and Newby, 2020; Basith et al., 2022).

FIPS	County name	Longitude	Latitude	PM2.5	CMR
1025	Clarke	-87.830772	31.676955	6.766443	379.421713
1061	Geneva	-85.839330	31.094869	8.254272	378.524698
1073	Jefferson	-86.896571	33.554343	10.825441	352.790427
1077	Lauderdale	-87.654117	34.901500	9.208783	332.594557
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- The treatment variable T, *i.e.*, the $PM_{2.5}$ level at each county, is a quantitative measure. In other words, it is *not a binary but continuous variable*!

The common causal estimands under a *binary* treatment are

- $\mathbb{E}[Y(t)]$ = mean counterfactual outcome when we set $T = t \in \{0, 1\}$.
- $\mathbb{E}[Y(1)] - \mathbb{E}[Y(0)]$ = average treatment effect.

▶ **Question:** What are the counterparts of the above estimands under a *continuous* treatment $T \in \mathcal{T} \subset \mathbb{R}$?

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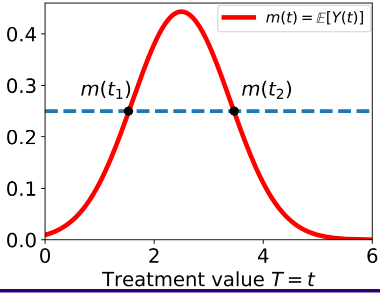
- $t \mapsto m(t) := \mathbb{E}\left[Y(t)\right]$ = (causal) dose-response curve.
- $t \mapsto \theta(t) := m'(t) = \frac{d}{dt}\mathbb{E}\left[Y(t)\right]$ = (causal) derivative effect curve.



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Challenges and Existing Approaches

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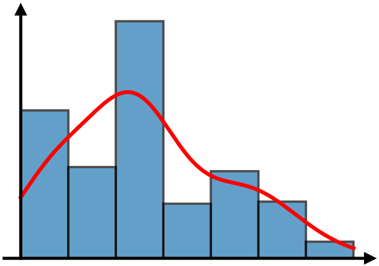
1. *Discretization:* Divide the range of T into bins and assign observations accordingly.



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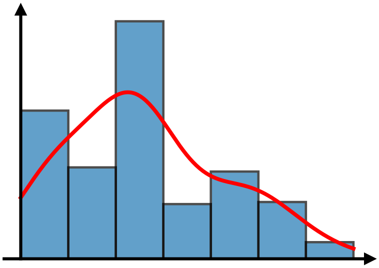

Pros: Allow direct applications of standard methods for discrete treatments (*e.g.*, the block-based diagnostics by Hirano and Imbens 2004; Bia and Mattei 2008).

Cons: Difficult to choose the cutoff points for binning.

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2. *Marginal Structural Model*: Impose parametric structural assumptions on $\mathbb{E}\left[Y(t)\right]$ (Robins et al., 2000; van der Laan and Robins, 2003; Neugebauer and van der Laan, 2007), *e.g.*,

$$m(t) = \mathbb{E}\left[Y(t)\right] = \alpha_1 + \alpha_2 \cdot t.$$

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Pros: Only need to estimate regression parameters α_1, α_2, which can achieve the parametric rate of convergence.

Cons: The parametric structural assumption could be violated!

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     $$\mathbb{E}\left[Y(T+\delta)\right] - \mathbb{E}\left[Y(T)\right] \quad \text{for some deterministic} \quad \delta > 0.$$

   - Average derivative effect (Härdle and Stoker, 1989; Powell et al., 1989; Newey and Stoker, 1993; Hines et al., 2023):

     $$\mathbb{E}\left[\theta(T)\right] = \mathbb{E}\left[\frac{\partial}{\partial t}\mathbb{E}\left(Y|T,S\right)\right], \quad \text{where } S \in \mathcal{S} \subset \mathbb{R}^d \text{ is a covariate vector.}$$

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Pros These new estimands may have more realistic interpretations in the actual context.

Cons They quantify only the overall causal effects, not those at a specific level of interest.

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Pros: Allow flexibility in estimating $m(t)$ and $\theta(t)$.

Cons: Require estimating nuisance functions and/or tuning (hyper)parameters.

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Cons Require estimating nuisance functions and/or tuning (hyper)parameters.

▶ Our works leverage **smoothness conditions** with **kernel smoothing** techniques.

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Assumption (Identification Condition)

1. *(Consistency)* $Y = Y(t)$ *whenever* $T = t \in \mathcal{T}$.

In randomized controlled trials (RCTs),

$$m(t) = \mathbb{E}\left[Y(t)\right] = \mathbb{E}\left(Y|T = t\right) \quad \text{and} \quad \theta(t) = \frac{d}{dt}\mathbb{E}\left[Y(t)\right] = \frac{d}{dt}\mathbb{E}\left(Y|T = t\right).$$

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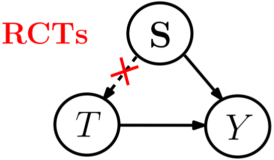
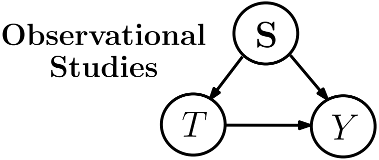
- Estimating $m(t)$ is to fit the regression function $t \mapsto \mathbb{E}(Y|T = t)$ on $\{(Y_i, T_i)\}_{i=1}^n$.
- Recovering $\theta(t)$ is a derivative estimation problem (Gasser and Müller, 1984).





[2]Some mild interchangeability assumptions are needed; see Theorem 1.1 in Shao (2003).

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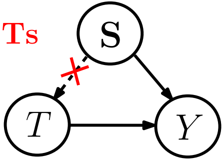
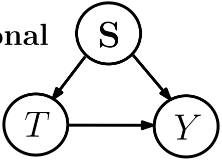


Observational Studies



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2. *(Ignorability)* $Y(t)$ *is conditionally independent of* T *given* $\boldsymbol{S}$ *for all* $t \in \mathcal{T}$.
3. *(**Positivity**) The conditional density satisfies* $p_{T|\boldsymbol{S}}(t|\boldsymbol{s}) \geq p_{\min} > 0$ *for all* $(t, \boldsymbol{s}) \in \mathcal{T} \times \mathcal{S}$.

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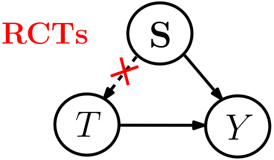

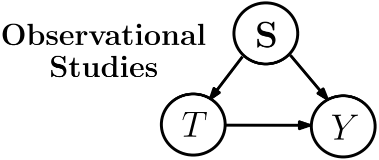

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- The positivity condition is required for $\mu(t, \boldsymbol{s}) = \mathbb{E}\left(Y|T = t, \boldsymbol{S} = \boldsymbol{s}\right)$ and $\frac{\partial}{\partial t}\mu(t, \boldsymbol{s}) = \frac{\partial}{\partial t}\mathbb{E}\left(Y|T = t, \boldsymbol{S} = \boldsymbol{s}\right)$ to be well-defined on $\mathcal{T} \times \mathcal{S}$.

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An Example of the Positivity Violation

Assumption (Positivity Condition)

There exists a constant $p_{\min} > 0$ *such that* $p_{T|S}(t|s) \geq p_{\min}$ *for all* $(t, s) \in \mathcal{T} \times \mathcal{S}$.

- Positivity is a very strong assumption with continuous treatments!

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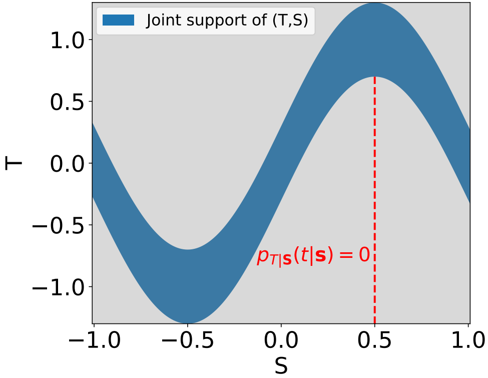
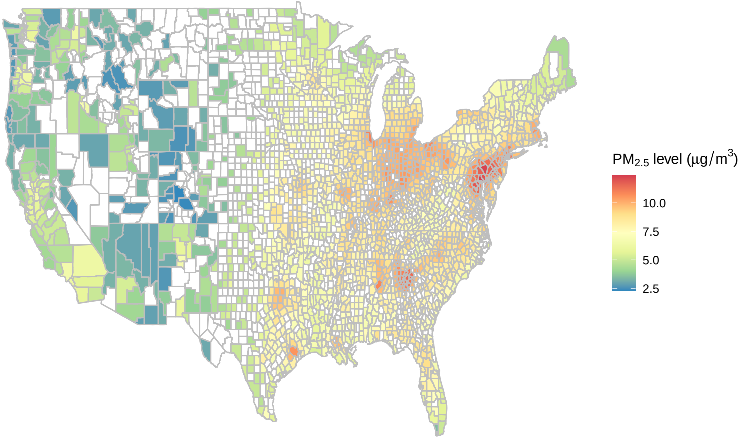
- Positivity is a very strong assumption with continuous treatments!

$$T = \sin(\pi S) + E, \quad E \sim \text{Uniform}[-0.3, 0.3], \quad S \sim \text{Uniform}[-1, 1], \quad \text{and} \quad E \perp\!\!\!\perp S.$$



Note that $p_{T|S}(t|s) = 0$ in the gray regions, and the positivity condition fails.

$PM_{2.5}$ Distribution at the County Level



Average $PM_{2.5}$ levels from 1990 to 2010 in $n = 2132$ counties.

- T is $PM_{2.5}$ level, and S consists of the county location and socioeconomic factors.
- Only one or several $PM_{2.5}$ levels are available per county in the dataset, and the positivity condition is violated!

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Under the positivity condition:

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Under the positivity condition:

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2. Propose our doubly robust (DR) estimator for $\theta(t)$.

$$\text{Regression Adjustment (RA)} \;+\; \text{Inverse Probability Weighting (IPW)} \begin{cases} \Rightarrow \\ \not\Rightarrow \end{cases} \text{DR.}$$

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Without the positivity condition:

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$$Y(t) = \bar{m}(t) + \eta(S) + \epsilon. \tag{1}$$

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5. Propose our bias-corrected IPW and DR estimators for $m(t)$ and $\theta(t)$.
   - Has a novel connection to nonparametric support and level set estimation problems.

Part I: Nonparametric Inference on $m(t)$ and $\theta(t)$ Under Positivity

This part is based on **Sections 2 and 3** in [1]:

[1] Y. Zhang and Y.-C. Chen (2025). **Doubly Robust Inference on Causal Derivative Effects for Continuous Treatments**. *arXiv:2501.06969*. https://arxiv.org/abs/2501.06969.

Assumption (Identification Conditions)

1. *(Consistency)* $Y = Y(t)$ *whenever* $T = t \in \mathcal{T}$.
2. *(Ignorability)* $Y(t)$ *is conditionally independent of* T *given* $\boldsymbol{S}$ *for all* $t \in \mathcal{T}$.
3. *(**Positivity**) The conditional density satisfies* $p_{T|\boldsymbol{S}}(t|\boldsymbol{s}) \geq p_{\min} > 0$ *for all* $(t, \boldsymbol{s}) \in \mathcal{T} \times \mathcal{S}$.

Given that $\mu(t, \boldsymbol{s}) = \mathbb{E}\left(Y | T = t, \boldsymbol{S} = \boldsymbol{s}\right)$, we have

RA or G-computation:
$$\begin{cases} m(t) = \mathbb{E}\left[Y(t)\right] = \mathbb{E}\left[\mu(t, \boldsymbol{S})\right], \\ \theta(t) = \frac{d}{dt}\mathbb{E}\left[Y(t)\right] = \frac{d}{dt}\mathbb{E}\left[\mu(t, \boldsymbol{S})\right] = \mathbb{E}\left[\frac{\partial}{\partial t}\mu(t, \boldsymbol{S})\right]. \end{cases}$$

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$$\textbf{RA or G-computation: } \begin{cases} m(t) = \mathbb{E}\left[Y(t)\right] = \mathbb{E}\left[\mu(t, \boldsymbol{S})\right], \\ \theta(t) = \frac{d}{dt}\mathbb{E}\left[Y(t)\right] = \frac{d}{dt}\mathbb{E}\left[\mu(t, \boldsymbol{S})\right] = \mathbb{E}\left[\frac{\partial}{\partial t}\mu(t, \boldsymbol{S})\right]. \end{cases}$$

$$\textbf{IPW: } \begin{cases} m(t) = \mathbb{E}\left[Y(t)\right] = \lim_{h \to 0} \mathbb{E}\left[\frac{Y}{p_{T|\boldsymbol{S}}(T|\boldsymbol{S})} \cdot \frac{1}{h} K\left(\frac{T-t}{h}\right)\right], \\ \theta(t) = \frac{d}{dt}\mathbb{E}\left[Y(t)\right] = \textbf{???}. \end{cases}$$

- $K: \mathbb{R} \to [0, \infty)$ is a kernel function, *e.g.*, $K(u) = \begin{cases} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) & \text{(Gaussian)}, \\ \frac{3}{4}(1 - u^2) \cdot \mathbb{1}_{\{|u| \leq 1\}} & \text{(Parabolic)}. \end{cases}$
- $h > 0$ is a smoothing bandwidth parameter.

Given the observed data $\{(Y_i, T_i, S_i)\}_{i=1}^n$, there are three main strategies for estimating

$$m(t) = \mathbb{E}[Y(t)] = \mathbb{E}[\mu(t, S)] = \lim_{h \to 0} \mathbb{E}\left[\frac{Y \cdot K\left(\frac{T-t}{h}\right)}{h \cdot p_{T|S}(T|S)}\right].$$

1. **RA Estimator** (Robins, 1986; Gill and Robins, 2001):

$$\widehat{m}_{\mathrm{RA}}(t) = \frac{1}{n}\sum_{i=1}^{n} \widehat{\mu}(t, S_i).$$

2. **IPW Estimator** (Hirano and Imbens, 2004; Imai and van Dyk, 2004):

$$\widehat{m}_{\mathrm{IPW}}(t) = \frac{1}{nh}\sum_{i=1}^{n} \frac{K\left(\frac{T_i-t}{h}\right)}{\widehat{p}_{T|S}(T_i|S_i)} \cdot Y_i.$$

3. **DR Estimator** (Kallus and Zhou, 2018; Colangelo and Lee, 2020):

$$\widehat{m}_{\mathrm{DR}}(t) = \frac{1}{nh}\sum_{i=1}^{n} \left\{\frac{K\left(\frac{T_i-t}{h}\right)}{\widehat{p}_{T|S}(T_i|S_i)} \cdot [Y_i - \widehat{\mu}(t, S_i)] + h \cdot \widehat{\mu}(t, S_i)\right\}.$$

To estimate $\theta(t) = \frac{d}{dt}\mathbb{E}\left[Y(t)\right] = \mathbb{E}\left[\frac{\partial}{\partial t}\mu(t,S)\right]$ from $\{(Y_i, T_i, S_i)\}_{i=1}^n$, we could also have three strategies:

1. **RA Estimator:**

$$\widehat{\theta}_{\mathrm{RA}}(t) = \frac{1}{n}\sum_{i=1}^n \widehat{\beta}(t, S_i) \quad \text{with} \quad \beta(t,s) = \frac{\partial}{\partial t}\mu(t,s).$$

Question: How to generalize the IPW form $m(t) = \lim_{h\to 0}\mathbb{E}\left[\frac{Y\cdot K\left(\frac{T-t}{h}\right)}{h\cdot p_{T|S}(T|S)}\right]$ to estimate $\theta(t)$?

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Question: How to generalize the IPW form $m(t) = \lim_{h\to 0}\mathbb{E}\left[\frac{Y\cdot K\left(\frac{T-t}{h}\right)}{h\cdot p_{T|S}(T|S)}\right]$ to estimate $\theta(t)$?

2. **IPW Estimator:** Inspired by the derivative estimator in Mack and Müller (1989), we propose

$$\widehat{\theta}_{\mathrm{IPW}}(t) = \frac{1}{n}\sum_{i=1}^n \frac{Y_i \cdot \left(\frac{T_i - t}{h}\right) K\left(\frac{T_i - t}{h}\right)}{h^2 \cdot \kappa_2 \cdot \widehat{p}_{T|S}(T_i|S_i)} \quad \text{with} \quad \kappa_2 = \int u^2 \cdot K(u)\, du.$$

The usual approach to construct a DR (or AIPW) estimator is as follows:

$$\widehat{m}_{\mathrm{RA}}(t) = \frac{1}{n}\sum_{i=1}^{n}\widehat{\mu}(t, S_i) \qquad \text{``+''} \qquad \widehat{m}_{\mathrm{IPW}}(t) = \frac{1}{nh}\sum_{i=1}^{n}\frac{K\left(\frac{T_i - t}{h}\right)}{\widehat{p}_{T|S}(T_i|S_i)} \cdot Y_i$$

$$\Longrightarrow \quad \widehat{m}_{\mathrm{DR}}(t) = \frac{1}{nh}\sum_{i=1}^{n}\frac{K\left(\frac{T_i - t}{h}\right)}{\widehat{p}_{T|S}(T_i|S_i)} \cdot \left[Y_i - \widehat{\mu}(t, S_i)\right] + \frac{1}{n}\sum_{i=1}^{n}\widehat{\mu}(t, S_i).$$

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$$\widehat{\theta}_{\mathrm{RA}}(t) = \frac{1}{n}\sum_{i=1}^{n} \widehat{\beta}(t, S_i) \qquad \text{“+”} \qquad \widehat{\theta}_{\mathrm{IPW}}(t) = \frac{1}{nh^2}\sum_{i=1}^{n} \frac{\left(\frac{T_i - t}{h}\right) K\left(\frac{T_i - t}{h}\right)}{\kappa_2 \cdot \widehat{p}_{T|S}(T_i|S_i)} \cdot Y_i \qquad \Longrightarrow$$

- $\widehat{\theta}_{\mathrm{AIPW},1}(t) = \frac{1}{nh^2}\sum_{i=1}^{n} \frac{\left(\frac{T_i - t}{h}\right) K\left(\frac{T_i - t}{h}\right)}{\kappa_2 \cdot \widehat{p}_{T|S}(T_i|S_i)} \left[Y_i - \widehat{\beta}(t, S_i)\right] + \frac{1}{n}\sum_{i=1}^{n} \widehat{\beta}(t, S_i)$;
- $\widehat{\theta}_{\mathrm{AIPW},2}(t) = \frac{1}{nh}\sum_{i=1}^{n} \frac{K\left(\frac{T_i - t}{h}\right)}{\widehat{p}_{T|S}(T_i|S_i)} \left[\frac{Y_i}{h \cdot \kappa_2}\left(\frac{T_i - t}{h}\right) - \widehat{\beta}(t, S_i)\right] + \frac{1}{n}\sum_{i=1}^{n} \widehat{\beta}(t, S_i)$; etc.

The usual approach to construct a DR (or AIPW) estimator is as follows:

$$\widehat{m}_{\mathrm{RA}}(t) = \frac{1}{n}\sum_{i=1}^{n} \widehat{\mu}(t, S_i) \qquad \text{“+”} \qquad \widehat{m}_{\mathrm{IPW}}(t) = \frac{1}{nh}\sum_{i=1}^{n} \frac{K\left(\frac{T_i - t}{h}\right)}{\widehat{p}_{T|S}(T_i|S_i)} \cdot Y_i$$

$$\Longrightarrow \quad \widehat{m}_{\mathrm{DR}}(t) = \frac{1}{nh}\sum_{i=1}^{n} \frac{K\left(\frac{T_i - t}{h}\right)}{\widehat{p}_{T|S}(T_i|S_i)} \cdot \left[Y_i - \widehat{\mu}(t, S_i)\right] + \frac{1}{n}\sum_{i=1}^{n} \widehat{\mu}(t, S_i).$$

$$\widehat{\theta}_{\mathrm{RA}}(t) = \frac{1}{n}\sum_{i=1}^{n} \widehat{\beta}(t, S_i) \qquad \text{“+”} \qquad \widehat{\theta}_{\mathrm{IPW}}(t) = \frac{1}{nh^2}\sum_{i=1}^{n} \frac{\left(\frac{T_i - t}{h}\right) K\left(\frac{T_i - t}{h}\right)}{\kappa_2 \cdot \widehat{p}_{T|S}(T_i|S_i)} \cdot Y_i \qquad \Longrightarrow$$

- $\widehat{\theta}_{\mathrm{AIPW},1}(t) = \frac{1}{nh^2}\sum_{i=1}^{n} \frac{\left(\frac{T_i - t}{h}\right) K\left(\frac{T_i - t}{h}\right)}{\kappa_2 \cdot \widehat{p}_{T|S}(T_i|S_i)} \left[Y_i - \widehat{\beta}(t, S_i)\right] + \frac{1}{n}\sum_{i=1}^{n} \widehat{\beta}(t, S_i)$;
- $\widehat{\theta}_{\mathrm{AIPW},2}(t) = \frac{1}{nh}\sum_{i=1}^{n} \frac{K\left(\frac{T_i - t}{h}\right)}{\widehat{p}_{T|S}(T_i|S_i)} \left[\frac{Y_i}{h \cdot \kappa_2}\left(\frac{T_i - t}{h}\right) - \widehat{\beta}(t, S_i)\right] + \frac{1}{n}\sum_{i=1}^{n} \widehat{\beta}(t, S_i)$; etc.

▶ **Remark:** All these AIPW estimators for $\theta(t)$ are, at best, **singly robust**!!

Doubly Robust Estimator for $\theta(t)$ Under Positivity

$$\widehat{\theta}_{\mathrm{RA}}(t) = \frac{1}{n}\sum_{i=1}^{n} \widehat{\beta}(t, S_i) \qquad \text{``+''} \qquad \widehat{\theta}_{\mathrm{IPW}}(t) = \frac{1}{nh^2}\sum_{i=1}^{n} \frac{\left(\frac{T_i - t}{h}\right) K\left(\frac{T_i - t}{h}\right)}{\kappa_2 \cdot \widehat{p}_{T|S}(T_i|S_i)} \cdot Y_i \qquad \Longrightarrow$$

$$\widehat{\theta}_{\mathrm{DR}}(t) = \underbrace{\frac{1}{nh^2}\sum_{i=1}^{n} \frac{\left(\frac{T_i - t}{h}\right) K\left(\frac{T_i - t}{h}\right)}{\kappa_2 \cdot \widehat{p}_{T|S}(T_i|S_i)} \left[Y_i - \widehat{\mu}(t, S_i) - (T_i - t) \cdot \widehat{\beta}(t, S_i)\right]}_{\text{New IPW component}} + \underbrace{\frac{1}{n}\sum_{i=1}^{n} \widehat{\beta}(t, S_i)}_{\text{RA component}}.$$

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The “New IPW component” leverages a local polynomial approximation to push the residual of the IPW component to (roughly) second order.

- Neyman orthogonality (Neyman, 1959; Chernozhukov et al., 2018) holds for this form of $\widehat{\theta}_{\mathrm{DR}}(t)$ as $h \to 0$.

Theorem (Theorem 1 in Zhang and Chen 2025)

Under some regularity assumptions and

1. $\widehat{\mu}, \widehat{\beta}, \widehat{p}_{T|S}$ *are estimated on a dataset independent of* $\{(Y_i, T_i, S_i)\}_{i=1}^n$;
2. *at least one of the model specification conditions hold:*
   - $\widehat{p}_{T|S}(t|s) \xrightarrow{P} \bar{p}_{T|S}(t|s) = p_{T|S}(t|s)$ (***conditional density model***),
   - $\widehat{\mu}(t,s) \xrightarrow{P} \bar{\mu}(t,s) = \mu(t,s)$ *and* $\widehat{\beta}(t,s) \xrightarrow{P} \bar{\beta}(t,s) = \beta(t,s)$ (***outcome model***);
3. $\sup_{|u-t|\leq h} \left|\left|\widehat{p}_{T|S}(u|S) - p_{T|S}(u|S)\right|\right|_{L_2} \left[\left|\left|\widehat{\mu}(t,S) - \mu(t,S)\right|\right|_{L_2} + h\left|\left|\widehat{\beta}(t,S) - \beta(t,S)\right|\right|_{L_2}\right] = o_P\left(\frac{1}{\sqrt{nh}}\right)$,

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3. $\sup_{|u-t|\leq h} \left|\left|\widehat{p}_{T|S}(u|S) - p_{T|S}(u|S)\right|\right|_{L_2} \left[\left|\left|\widehat{\mu}(t,S) - \mu(t,S)\right|\right|_{L_2} + h\left|\left|\widehat{\beta}(t,S) - \beta(t,S)\right|\right|_{L_2}\right] = o_P\left(\frac{1}{\sqrt{nh}}\right)$,

we prove that

- $\sqrt{nh^3}\left[\widehat{\theta}_{\mathrm{DR}}(t) - \theta(t)\right] = \frac{1}{\sqrt{n}}\sum_{i=1}^n \phi_{h,t}\left(Y_i, T_i, S_i; \bar{\mu}, \bar{\beta}, \bar{p}_{T|S}\right) + o_P(1)$.
- $\sqrt{nh^3}\left[\widehat{\theta}_{\mathrm{DR}}(t) - \theta(t) - h^2 B_\theta(t)\right] \xrightarrow{d} \mathcal{N}\left(0, V_\theta(t)\right)$.

Statistical Inference on $\theta(t)$

An asymptotically valid inference on $\theta(t) = \frac{d}{dt}\mathbb{E}\left[Y(t)\right]$ can be conducted through

$$\sqrt{nh^3}\left[\widehat{\theta}_{\mathrm{DR}}(t) - \theta(t) - h^2 B_{\theta}(t)\right] \xrightarrow{d} \mathcal{N}\left(0, V_{\theta}(t)\right).$$

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1. We estimate $V_{\theta}(t) = \mathbb{E}\left[\phi_{h,t}^2\left(Y, T, S; \bar{\mu}, \bar{\beta}, \bar{p}_{T|S}\right)\right]$ with

$$\phi_{h,t}\left(Y, T, S; \bar{\mu}, \bar{\beta}, \bar{p}_{T|S}\right) = \frac{\left(\frac{T-t}{h}\right) K\left(\frac{T-t}{h}\right)}{\sqrt{h}\cdot\kappa_2\cdot\bar{p}_{T|S}(T|S)}\cdot\left[Y - \bar{\mu}(t,S) - (T-t)\cdot\bar{\beta}(t,S)\right]$$

by $\widehat{V}_{\theta}(t) = \frac{1}{n}\sum_{i=1}^{n}\phi_{h,t}^2\left(Y, T, S; \widehat{\mu}, \widehat{\beta}, \widehat{p}_{T|S}\right)$.

Statistical Inference on $\theta(t)$

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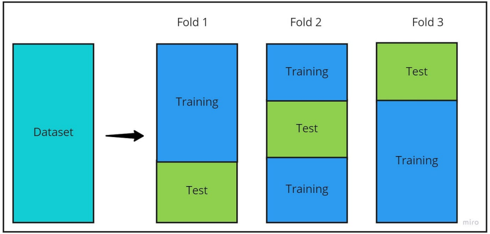
2. $\widehat{\mu}, \widehat{\beta}, \widehat{p}_{T|S}$ can be estimated via sample-splitting or cross-fitting.



Statistical Inference on $\theta(t)$

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   by $\widehat{V}_\theta(t) = \frac{1}{n}\sum_{i=1}^{n}\phi_{h,t}^2\left(Y, T, S; \widehat{\mu}, \widehat{\beta}, \widehat{p}_{T|S}\right)$.

2. $\widehat{\mu}, \widehat{\beta}, \widehat{p}_{T|S}$ can be estimated via sample-splitting or cross-fitting.

3. The explicit form of $B_\theta(t)$ is complicated, but $h^2 B_\theta(t)$ is asymptotically negligible when $h = O\left(n^{-\frac{1}{5}}\right)$.
   - This order aligns with the outputs from usual bandwidth selection methods (Wand and Jones, 1994; Wasserman, 2006).

Nonparametric Efficiency Guarantee for $\widehat{\theta}_{\mathrm{DR}}(t)$

Question:[3] Do we have a nonparametric efficiency lower bound for $\widehat{\theta}_{\mathrm{DR}}(t)$?

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- $t \mapsto \theta(t) := \Psi(\mathrm{P}_0)(t)$ is *not* pathwise differentiable (Bickel et al., 1998; Hirano and Porter, 2012; Luedtke and van der Laan, 2016):

$$\forall t \in \mathcal{T}, \quad \exists \{\mathrm{P}_\epsilon : \epsilon \in \mathbb{R}\} \quad \text{s.t.} \quad \lim_{\epsilon \to 0} \frac{\Psi(\mathrm{P}_\epsilon)(t) - \Psi(\mathrm{P}_0)(t)}{\epsilon} \quad \text{does not exist.}$$

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- For a fixed $h > 0$, the smooth functional $\Phi(\mathrm{P}_0)(t) := \mathbb{E}\left[\frac{Y \cdot \left(\frac{T-t}{h}\right) K\left(\frac{T-t}{h}\right)}{h^2 \cdot \kappa_2 \cdot p_{T|S}(T|S)}\right]$ is pathwise differentiable (van der Laan et al., 2018; Takatsu and Westling, 2024).

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- For a fixed $h > 0$, the smooth functional $\Phi(\mathrm{P}_0)(t) := \mathbb{E}\left[\frac{Y \cdot \left(\frac{T-t}{h}\right) K\left(\frac{T-t}{h}\right)}{h^2 \cdot \kappa_2 \cdot p_{T|S}(T|S)}\right]$ is pathwise differentiable (van der Laan et al., 2018; Takatsu and Westling, 2024).
- Up to a shrinking bias $O(h^2)$, the efficient influence function for $\Phi(\mathrm{P}_0)(t)$ leads to

$$\widehat{\theta}_{\mathrm{EIF}}(t) = \frac{1}{nh^2} \sum_{i=1}^n \frac{\left(\frac{T_i - t}{h}\right) K\left(\frac{T_i - t}{h}\right)}{\kappa_2 \cdot \widehat{p}_{T|S}(T_i|S_i)} \left[Y_i - \widehat{\mu}(T_i, S_i)\right] + \frac{1}{n} \sum_{i=1}^n \widehat{\beta}(t, S_i).$$

► The asymptotic variances of $\widehat{\theta}_{\mathrm{DR}}(t)$ and $\widehat{\theta}_{\mathrm{EIF}}(t)$ are the same (or differing by $O(h^2)$)!

[3] I acknowledge Ted Westling and Aaron Hudson for pointing out this direction.

Part II: Nonparametric Inference on $m(t)$ and $\theta(t)$ Without Positivity

This part is based on **Sections 4 and 5** in [1] and **Sections 2, 3, and 4** in [2]:

[1] Y. Zhang and Y.-C. Chen (2025). **Doubly Robust Inference on Causal Derivative Effects for Continuous Treatments**. *arXiv:2501.06969*. https://arxiv.org/abs/2501.06969.

[2] Y. Zhang, Y.-C. Chen, and A. Giessing (2024). **Nonparametric Inference on Dose-Response Curves Without the Positivity Condition**. *arXiv:2405.09003*. https://arxiv.org/abs/2405.09003.

Why Do We Need Positivity?

Assumption (Identification Conditions)

1. *(Consistency)* $Y = Y(t)$ *whenever* $T = t \in \mathcal{T}$.
2. *(Ignorability or Unconfoundedness)* $Y(t) \perp\!\!\!\perp T \,\big|\, S$ *for all* $t \in \mathcal{T}$.
3. *(**Positivity**)* $p_{T|S}(t|s) \geq p_{\min} > 0$ *for all* $(t, s) \in \mathcal{T} \times \mathcal{S}$.

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Assumption (Identification Conditions)

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▶ **Identification Issue:** Without positivity, $\mu(t, \boldsymbol{s}) = \mathbb{E}\left(Y | T = t, \boldsymbol{S} = \boldsymbol{s}\right)$ is *not well-defined* outside the support $\mathcal{J} \subset \mathcal{T} \times \mathcal{S}$ of the joint density $p(t, \boldsymbol{s})$.

Key Example: Additive Confounding Model

Consider the additive confounding model (Paciorek, 2010; Schnell and Papadogeorgou, 2020; Gilbert et al., 2023):

$$Y(t) = \bar{m}(t) + \eta(S) + \epsilon \quad \text{with} \quad \mathbb{E}(\epsilon) = 0 \quad \text{and} \quad \mathrm{Var}(\epsilon) > 0. \tag{2}$$

- $\bar{m} : \mathcal{T} \to \mathbb{R}, \eta : \mathcal{S} \to \mathbb{R}$ are unknown functions, while $\epsilon \in \mathbb{R}$ is exogenous.
- $m(t) = \mathbb{E}\left[Y(t)\right] = \bar{m}(t) + \mathbb{E}\left[\eta(S)\right]$ and $\theta(t) = m'(t) = \frac{d}{dt}\mathbb{E}\left[Y(t)\right] = \bar{m}'(t)$.

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▶ **Identification of $\theta(t)$:** Under model (2) and consistency, we have

$$\theta(t) = \mathbb{E}\left[\frac{\partial}{\partial t}\mu(t,S)\Big| T = t\right] := \theta_C(t) \quad \text{and} \quad \mathbb{E}(Y) = \mathbb{E}\left[m(T)\right].$$

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▶ **Identification of $m(t)$:** By the fundamental theorem of calculus,

$$m(t) = \mathbb{E}\left[Y + \int_{u=T}^{u=t} \theta_C(u)\, du\right] = \mathbb{E}(Y) + \mathbb{E}\left\{\int_{u=T}^{u=t} \mathbb{E}\left[\frac{\partial}{\partial t}\mu(T, S)\Big| T = u\right] du\right\} \text{ for any } t \in \mathcal{T}.$$

▶ **Drawback of (2):** The treatment effect is homogeneous for any $S = s \in \mathcal{S}$.

Estimation of $m(t)$ and $\theta(t)$ Without Positivity

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$$\widehat{m}_{\mathrm{C,RA}}(t) = \frac{1}{n}\sum_{i=1}^{n}\left[Y_i + \int_{\tilde{t}=T_i}^{\tilde{t}=t} \widehat{\theta}_{\mathrm{C,RA}}(\tilde{t})\, d\tilde{t}\right] \quad \text{and} \quad \widehat{\theta}_{\mathrm{C,RA}}(t) = \int \widehat{\beta}(t, s)\, d\widehat{F}_{S|T}(s|t).$$

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- Establish the consistency of nonparametric bootstrap for $\widehat{m}_{\mathrm{C,RA}}(t)$ and $\widehat{\theta}_{\mathrm{C,RA}}(t)$.

Estimation Biases of IPW Estimators Without Positivity

Question: How about IPW and DR estimators of $\theta(t)$ (and $m(t)$) without positivity?

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Proposition (Proposition 2 in Zhang and Chen 2025)

$$\lim_{h\to 0} \mathbb{E}\left[\tilde{m}_{\mathrm{IPW}}(t)\right] = \bar{m}(t)\cdot\rho(t) + \omega(t) \neq m(t), \qquad \textit{with} \quad \rho(t) = \mathbb{P}\left(S \in \mathcal{S}(t)\right),$$

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► **Key Issue:** The conditional support $\mathcal{S}(t)$ of $p_{S|T}(s|t)$ and the marginal support $\mathcal{S}$ of $p_S(s)$ are different under the violations of positivity!!

Bias-Corrected IPW Estimator for $\theta(t)$

$$\lim_{h\to 0} \mathbb{E}\left[\widetilde{\theta}_{\mathrm{IPW}}(t)\right] = \lim_{h\to 0} \mathbb{E}\left[\frac{Y\left(\frac{T-t}{h}\right) K\left(\frac{T-t}{h}\right)}{h^2 \cdot \kappa_2 \cdot p_{T|S}(T|S)}\right] = \begin{cases} \bar{m}'(t) \cdot \rho(t) \\ \infty \end{cases} \neq \theta(t),$$

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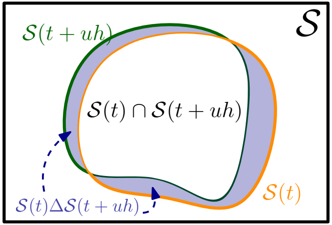
1. We first want to disentangle $\theta(t) = \bar{m}'(t)$ from the bias term:

$$\mathbb{E}\left[\frac{Y\cdot\left(\frac{T-t}{h}\right) K\left(\frac{T-t}{h}\right)\cdot p_{S|T}(S|t)}{h^2\cdot\kappa_2\cdot p_{T|S}(T|S)\cdot p_S(S)}\right] = \bar{m}'(t) + O(h^2)$$

$$+ \underbrace{\int_{\mathbb{R}} \mathbb{E}\left\{\left[\bar{m}(t+uh)+\eta(S)\right]\left[\mathbb{1}_{\{S\in\mathcal{S}(t+uh)\setminus\mathcal{S}(t)\}} - \mathbb{1}_{\{S\in\mathcal{S}(t)\setminus\mathcal{S}(t+uh)\}}\right] \Big| T=t\right\} u\cdot K(u)\,du}_{\text{Non-vanishing Bias}}.$$



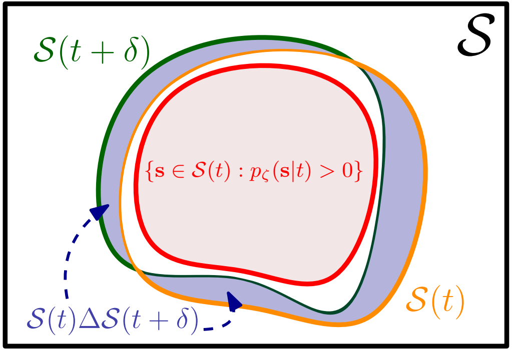
Bias-Corrected IPW Estimator for $\theta(t)$

$$\mathbb{E}\left[\frac{Y\cdot\left(\frac{T-t}{h}\right)K\left(\frac{T-t}{h}\right)p_{\boldsymbol{S}|T}(\boldsymbol{S}|t)}{h^2\cdot\kappa_2\cdot p_{T|\boldsymbol{S}}(T|\boldsymbol{S})\cdot p_{\boldsymbol{S}}(\boldsymbol{S})}\right]=\bar{m}'(t)+O(h^2)+\text{“Non-vanishing Bias”}.$$

$$\mathbb{E}\left[\frac{Y\cdot\left(\frac{T-t}{h}\right)K\left(\frac{T-t}{h}\right)p_{S|T}(S|t)}{h^2\cdot\kappa_2\cdot p_{T|S}(T|S)\cdot p_S(S)}\right] = \bar{m}'(t) + O(h^2) + \text{“Non-vanishing Bias”}.$$

2. We replace $p_{S|T}(s|t)$ with a ζ-interior conditional density $p_\zeta(s|t)$ so that

$$\{s\in\mathcal{S}(t) : p_\zeta(s|t)>0\}\subset\mathcal{S}(t+\delta) \quad \text{for any} \quad \delta\in[-h,h].$$



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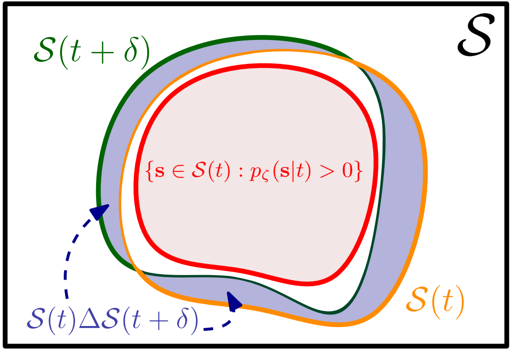

Now, we have that

$$\mathbb{E}\left[\frac{Y \cdot \left(\frac{T-t}{h}\right) K\left(\frac{T-t}{h}\right) p_\zeta(S|t)}{h^2 \cdot \kappa_2 \cdot p_{T|S}(T|S) \cdot p_S(S)}\right] = \bar{m}'(t) + O(h^2).$$

Question: How can we find a ζ-interior conditional density $p_\zeta(\boldsymbol{s}|t)$?

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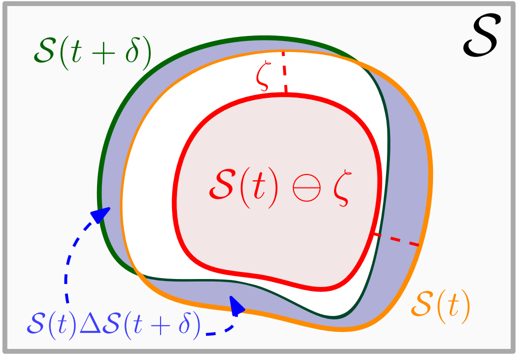
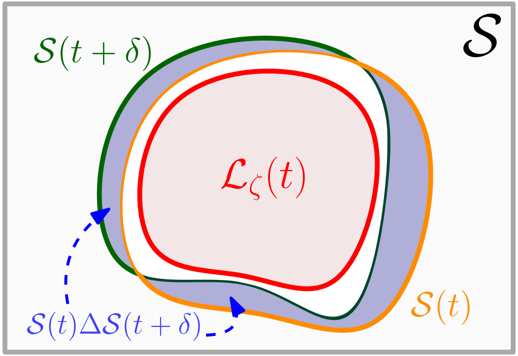
Support shrinking approach



Level set approach



$$\mathcal{S}(t)\ominus\zeta = \left\{\mathbf{s}\in\mathcal{S}(t) : \inf_{\mathbf{x}\in\partial\mathcal{S}(t)} ||\mathbf{s}-\mathbf{x}||_2 \geq \zeta\right\},$$

$$p_\zeta(\mathbf{s}|t) = \frac{p_{S|T}(\mathbf{s}|t)\cdot \mathbb{1}_{\{\mathbf{s}\in\mathcal{S}(t)\ominus\zeta\}}}{\int_{\mathcal{S}(t)\ominus\zeta} p_{S|T}(\mathbf{s}_1|t)\, d\mathbf{s}_1}.$$

$$\mathcal{L}_\zeta(t) = \left\{\mathbf{s}\in\mathcal{S}(t) : p_{S|T}(\mathbf{s}|t)\geq\zeta\right\},$$

$$p_\zeta(\mathbf{s}|t) = \frac{p_{S|T}(\mathbf{s}|t)\cdot \mathbb{1}_{\{\mathbf{s}\in\mathcal{L}_\zeta(t)\}}}{\int_{\mathcal{L}_\zeta(t)} p_{S|T}(\mathbf{s}_1|t)\, d\mathbf{s}_1}.$$

▶ **Bias-Corrected IPW Estimator Without Positivity:**

$$\widehat{\theta}_{\mathrm{C,IPW}}(t) = \frac{1}{nh^2}\sum_{i=1}^{n} \frac{Y_i \cdot \left(\frac{T_i - t}{h}\right) K\left(\frac{T_i - t}{h}\right) \widehat{p}_{\zeta}(\boldsymbol{S}_i|t)}{\kappa_2 \cdot \widehat{p}(T_i, \boldsymbol{S}_i)},$$

- $\widehat{p}(t, \boldsymbol{s}), \widehat{p}_{\zeta}(\boldsymbol{s}|t)$ are estimators of $p(t, \boldsymbol{s}), p_{\zeta}(\boldsymbol{s}|t)$ and $\zeta = 0.5 \cdot \max\left\{\widehat{p}_{S|T}(\boldsymbol{S}_i|t) : i = 1, ..., n\right\}$.

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▶ **Bias-Corrected DR Estimator Without Positivity:**

$$\widehat{\theta}_{\mathrm{C,DR}}(t) = \underbrace{\frac{1}{nh^2} \sum_{i=1}^{n} \frac{\left(\frac{T_i - t}{h}\right) K\left(\frac{T_i - t}{h}\right) \widehat{p}_{\zeta}(\boldsymbol{S}_i|t)}{\kappa_2 \cdot \widehat{p}(T_i, \boldsymbol{S}_i)} \left[Y_i - \widehat{\mu}(t, \boldsymbol{S}_i) - (T_i - t) \cdot \widehat{\beta}(t, \boldsymbol{S}_i)\right]}_{\text{IPW component}}$$

$$+ \underbrace{\int \widehat{\beta}(t, \boldsymbol{s}) \cdot \widehat{p}_{\zeta}(\boldsymbol{s}|t)\, d\boldsymbol{s}}_{\text{RA component}}.$$

Bias-Corrected IPW and DR Estimators of $\theta(t)$

▶ **Bias-Corrected IPW Estimator Without Positivity:**

$$\widehat{\theta}_{\mathrm{C,IPW}}(t) = \frac{1}{nh^2}\sum_{i=1}^{n} \frac{Y_i \cdot \left(\frac{T_i - t}{h}\right) K\left(\frac{T_i - t}{h}\right) \widehat{p}_\zeta(\boldsymbol{S}_i|t)}{\kappa_2 \cdot \widehat{p}(T_i, \boldsymbol{S}_i)},$$

- $\widehat{p}(t, \boldsymbol{s}), \widehat{p}_\zeta(\boldsymbol{s}|t)$ are estimators of $p(t, \boldsymbol{s}), p_\zeta(\boldsymbol{s}|t)$ and $\zeta = 0.5 \cdot \max\left\{\widehat{p}_{S|T}(\boldsymbol{S}_i|t) : i = 1, ..., n\right\}$.

▶ **Bias-Corrected DR Estimator Without Positivity:**

$$\widehat{\theta}_{\mathrm{C,DR}}(t) = \underbrace{\frac{1}{nh^2}\sum_{i=1}^{n} \frac{\left(\frac{T_i - t}{h}\right) K\left(\frac{T_i - t}{h}\right) \widehat{p}_\zeta(\boldsymbol{S}_i|t)}{\kappa_2 \cdot \widehat{p}(T_i, \boldsymbol{S}_i)} \left[Y_i - \widehat{\mu}(t, \boldsymbol{S}_i) - (T_i - t) \cdot \widehat{\beta}(t, \boldsymbol{S}_i)\right]}_{\text{IPW component}}$$

$$+ \underbrace{\int \widehat{\beta}(t, \boldsymbol{s}) \cdot \widehat{p}_\zeta(\boldsymbol{s}|t)\, d\boldsymbol{s}}_{\text{RA component}}.$$

▶ **Remark:** Practically, the RA estimators $\widehat{\theta}_{\mathrm{C,RA}}(t)$ and $\widehat{m}_{\mathrm{C,RA}}(t)$ are recommended!

Theorem (Theorem 5 in Zhang and Chen 2025)

Under some regularity assumptions and

1. $\widehat{\mu}, \widehat{\beta}, \widehat{p}, \widehat{p}_\zeta$ *are estimated on a dataset independent of* $\{(Y_i, T_i, \boldsymbol{S}_i)\}_{i=1}^n$;
2. $\sqrt{nh}\,||\widehat{p}_\zeta(\boldsymbol{S}|t) - \bar{p}_\zeta(\boldsymbol{S}|t)||_{L_2} = o_P(1)$ *with* $\widehat{p}_\zeta(\boldsymbol{s}|t) \xrightarrow{P} \bar{p}_\zeta(\boldsymbol{s}|t)$;
3. *at least one of the model specification conditions hold:*
    - $\widehat{p}(t,\boldsymbol{s}) \xrightarrow{P} \bar{p}(t,\boldsymbol{s}) = p(t,\boldsymbol{s})$ *(joint density model),*
    - $\widehat{\mu}(t,\boldsymbol{s}) \xrightarrow{P} \bar{\mu}(t,\boldsymbol{s}) = \mu(t,\boldsymbol{s})$ *and* $\widehat{\beta}(t,\boldsymbol{s}) \xrightarrow{P} \bar{\beta}(t,\boldsymbol{s}) = \beta(t,\boldsymbol{s})$ *(outcome model);*
4. $\sup_{|u-t|\le h} ||\widehat{p}(u,\boldsymbol{S}) - p(u,\boldsymbol{S})||_{L_2}\left[||\widehat{\mu}(t,\boldsymbol{S}) - \mu(t,\boldsymbol{S})||_{L_2} + h\left|\left|\widehat{\beta}(t,\boldsymbol{S}) - \beta(t,\boldsymbol{S})\right|\right|_{L_2}\right] = o_P\left(\frac{1}{\sqrt{nh}}\right)$,

we prove that

- $\sqrt{nh^3}\left[\widehat{\theta}_{\mathrm{C,DR}}(t) - \theta(t)\right] = \frac{1}{\sqrt{n}}\sum_{i=1}^n \phi_{\mathrm{C},h,t}\left(Y_i, T_i, \boldsymbol{S}_i; \bar{\mu}, \bar{\beta}, \bar{p}_{T|\boldsymbol{S}}\right) + o_P(1)$.
- $\sqrt{nh^3}\left[\widehat{\theta}_{\mathrm{C,DR}}(t) - \theta(t) - h^2 \cdot B_{\mathrm{C},\theta}(t)\right] \xrightarrow{d} \mathcal{N}\left(0, V_{\mathrm{C},\theta}(t)\right)$.

Application: $PM_{2.5}$ on CMR

This part is based on **Section 5.3** in [2]:

[2] Y. Zhang, Y.-C. Chen, and A. Giessing (2024). **Nonparametric Inference on Dose-Response Curves Without the Positivity Condition**. *arXiv:2405.09003*. https://arxiv.org/abs/2405.09003.

All the code and data are available at

https://github.com/zhangyk8/npDoseResponse/tree/main.

Python Package: npDoseResponse and R Package: npDoseResponse.

$PM_{2.5}$ and CMRs Data Recap

FIPS	County name	Longitude	Latitude	PM2.5	CMR
1025	Clarke	-87.830772	31.676955	6.766443	379.421713
1061	Geneva	-85.839330	31.094869	8.254272	378.524698
1073	Jefferson	-86.896571	33.554343	10.825441	352.790427
1077	Lauderdale	-87.654117	34.901500	9.208783	332.594557
5085	Lonoke	-91.887917	34.754412	8.213144	365.061085
8045	Garfield	-107.903621	39.599420	2.601772	250.781477

1. The dataset (Wyatt et al., 2020a,b) contains the average annual CMRs (Y) and $PM_{2.5}$ levels (T) across $n = 2132$ U.S. counties over 1990-2010.

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   - 2 spatial confounders: latitude and longitude of each county.
   - 8 county-level socioeconomic factors acquired from the US census.

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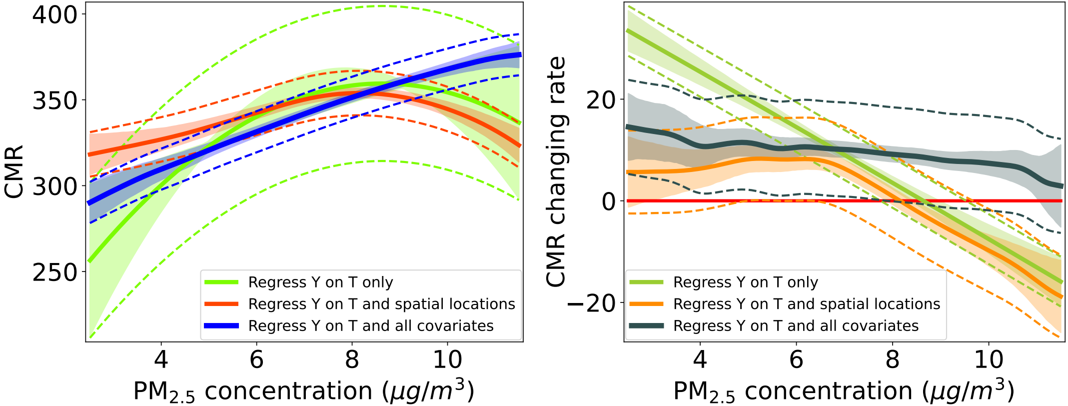
2. The covariate vector $\boldsymbol{S} \in \mathbb{R}^{10}$ consists of two parts:
   - 2 spatial confounders: latitude and longitude of each county.
   - 8 county-level socioeconomic factors acquired from the US census.
3. Focus on the values of $PM_{2.5}$ between 2.5 $\mu g/m^3$ and 11.5 $\mu g/m^3$ to avoid boundary effects (Takatsu and Westling, 2024).

Effect of $PM_{2.5}$ on the Cardiovascular Mortality Rate (CMR)



Shaded areas: 95% pointwise confidence intervals; **Regions between dashed lines:** 95% uniform confidence bands.

- We compare three models:
  1. Regress Y on T alone via local quadratic regression.
  2. Regress Y on T with spatial locations.
  3. Regress Y on T with both spatial and socioeconomic covariates.

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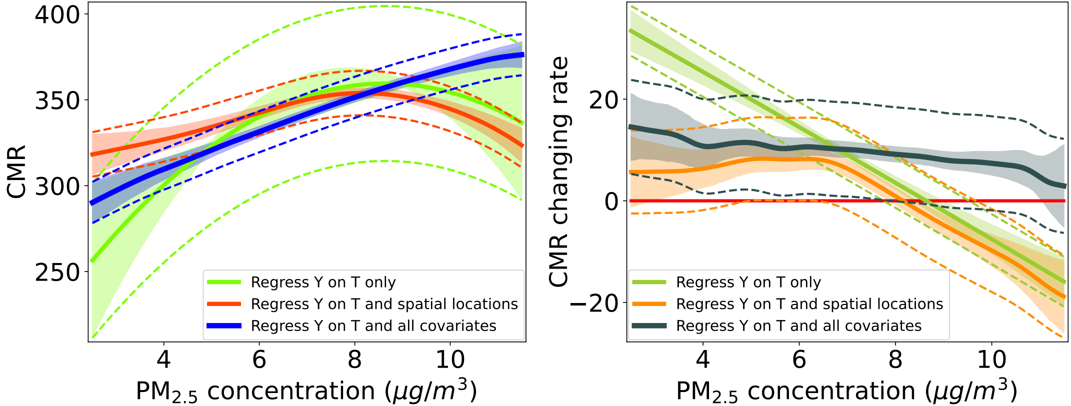

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- We compare three models:
  1. Regress Y on T alone via local quadratic regression.
  2. Regress Y on T with spatial locations.
  3. Regress Y on T with both spatial and socioeconomic covariates.
- For model 3, the increasing trends are **significant** when $PM_{2.5} < 8\,\mu g/m^3$.

Discussion

We study nonparametric inference on $m(t) = \mathbb{E}\left[Y(t)\right]$ and $\theta(t) = \frac{d}{dt}\mathbb{E}\left[Y(t)\right]$, $t \in \mathcal{T} \subset \mathbb{R}$.

We study nonparametric inference on $m(t) = \mathbb{E}[Y(t)]$ and $\theta(t) = \frac{d}{dt}\mathbb{E}[Y(t)]$, $t \in \mathcal{T} \subset \mathbb{R}$.

1. **Under the positivity condition:**
   - We propose $\widehat{\theta}_{\mathrm{DR}}(t)$ with standard nonparametric consistency and efficiency guarantee:

$$\sqrt{nh^3}\left[\widehat{\theta}_{\mathrm{DR}}(t) - \theta(t) - h^2 B_\theta(t)\right] \stackrel{d}{\to} \mathcal{N}\left(0, V_\theta(t)\right).$$

Punchlines of Today's Talk

We study nonparametric inference on $m(t) = \mathbb{E}\left[Y(t)\right]$ and $\theta(t) = \frac{d}{dt}\mathbb{E}\left[Y(t)\right]$, $t \in \mathcal{T} \subset \mathbb{R}$.

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2. **Without the positivity condition:**
   - Our key technique relies on two pillars in calculus:

$$\underbrace{\theta(t) = \mathbb{E}\left[\frac{\partial}{\partial t}\mu(t,S)\Big| T = t\right]}_{\textbf{Differentiation}} \quad \text{and} \quad \underbrace{m(t) = \mathbb{E}\left[Y + \int_{u=T}^{u=t} \theta(u)\, du\right]}_{\textbf{Integration}}.$$

Punchlines of Today's Talk

We study nonparametric inference on $m(t) = \mathbb{E}\left[Y(t)\right]$ and $\theta(t) = \frac{d}{dt}\mathbb{E}\left[Y(t)\right], t \in \mathcal{T} \subset \mathbb{R}$.

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   - Our bias-corrected IPW and DR estimators reveal interesting connections to nonparametric level set estimation problems (Bonvini et al., 2023):

$$\textit{Causal Inference} \quad \Longleftrightarrow \quad \textit{Geometric Data Analysis}.$$

1. **Debiasing Doubly Robust Estimators:** Can we debias our DR estimators $\widehat{\theta}_{\mathrm{DR}}(t)$ and $\widehat{\theta}_{\mathrm{C,DR}}(t)$ through explicit bias estimation (Calonico et al., 2018; Cheng and Chen, 2019; Takatsu and Westling, 2024) or calibration (van der Laan et al., 2024)?

Ongoing Works and Future Directions

1. **Debiasing Doubly Robust Estimators:** Can we debias our DR estimators $\widehat{\theta}_{\mathrm{DR}}(t)$ and $\widehat{\theta}_{\mathrm{C,DR}}(t)$ through explicit bias estimation (Calonico et al., 2018; Cheng and Chen, 2019; Takatsu and Westling, 2024) or calibration (van der Laan et al., 2024)?

2. **Violation of Ignorability:** Can we conduct sensitivity analysis on unmeasured confounding (Chernozhukov et al., 2022a)?

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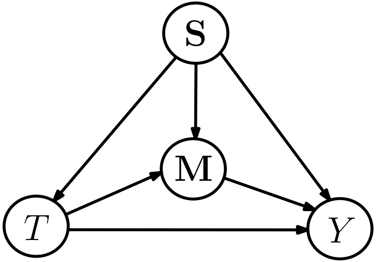
2. **Violation of Ignorability:** Can we conduct sensitivity analysis on unmeasured confounding (Chernozhukov et al., 2022a)?

3. **Mediation Analysis:** Can we generalize our strategies for the estimation of direct and indirect causal effects (Huber et al., 2020; Xu et al., 2021)?



Thank you!

More details can be found in

[1] Y. Zhang, Y.-C. Chen, and A. Giessing. Nonparametric Inference on Dose-Response Curves Without the Positivity Condition. *arXiv preprint*, 2024. https://arxiv.org/abs/2405.09003.
[2] Y. Zhang and Y.-C. Chen. Doubly Robust Inference on Causal Derivative Effects for Continuous Treatments. *arXiv preprint*, 2025. https://arxiv.org/abs/2501.06969.

All the code and data are available at
https://github.com/zhangyk8/npDoseResponse/tree/main.

Python Package: npDoseResponse and R Package: npDoseResponse.

Check out my other works at https://zhangyk8.github.io/.

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1. **Order q (Partial) Local Polynomial Regression** (Fan and Gijbels, 1996): Let $\widehat{\boldsymbol{\beta}}(t, \boldsymbol{s}) \in \mathbb{R}^{q+1}$ and $\widehat{\boldsymbol{\alpha}}(t, \boldsymbol{s}) \in \mathbb{R}^d$ be the minimizer of

$$\underset{(\boldsymbol{\beta},\boldsymbol{\alpha})^T \in \mathbb{R}^{q+1+d}}{\arg\min} \sum_{i=1}^{n} \left[Y_i - \sum_{j=0}^{q} \beta_j (T_i - t)^q - \sum_{\ell=1}^{d} \alpha_\ell (S_{i,\ell} - s_\ell) \right]^2 K_T\left(\frac{T_i - t}{h}\right) K_S\left(\frac{\boldsymbol{S}_i - \boldsymbol{s}}{b}\right).$$

   - $K_T : \mathbb{R} \to [0, \infty)$, $K_S : \mathbb{R}^d \to [0, \infty)$ are two symmetric kernel functions, and $h, b > 0$ are smoothing bandwidth parameters.
   - The second component $\widehat{\beta}_2(t, \boldsymbol{s}) := \widehat{\beta}(t, \boldsymbol{s})$ is a consistent estimator of $\beta(t, \boldsymbol{s}) = \frac{\partial}{\partial t}\mu(t, \boldsymbol{s})$.

2. **Nadaraya-Watson Conditional CDF Estimator for $F_{S|T}(\boldsymbol{s}|t)$** (Hall et al., 1999):

$$\widehat{F}_{S|T}(\boldsymbol{s}|t) = \widehat{P}_{\hbar}(\boldsymbol{s}|t) = \frac{\sum_{i=1}^{n} \mathbb{1}_{\{\boldsymbol{S}_i \leq \boldsymbol{s}\}} \cdot \bar{K}_T\left(\frac{T_i - t}{\hbar}\right)}{\sum_{j=1}^{n} \bar{K}_T\left(\frac{T_j - t}{\hbar}\right)}.$$

   - $\bar{K}_T : \mathbb{R} \to [0, \infty)$ is a kernel function and $\hbar > 0$ is its smoothing bandwidth parameter.

3 **Kernel Smoothing Methods for Estimating $p_{T|S}(t|s)$:**

- *Method 1 (Kernel density estimation on residuals):* If $T = g_S(S) + g_E(E)$ with $\mathbb{E}\left[g_E(E)|S\right] = 0$, then we can estimate $g_S(s) = \mathbb{E}\left(T|S = s\right)$ via any machine learning method to obtain

$$\widehat{p}_{T|S}(t|s) = \frac{1}{nh_e}\sum_{i=1}^{n} K_e\left[\frac{t - \widehat{g}_S(s) - (T_i - \widehat{g}_S(S_i))}{h_e}\right],$$

where $K_e : \mathbb{R} \to [0, \infty)$ is a kernel function and $h_e > 0$ is a bandwidth parameter.

- *Method 2 (Regression on kernel-smoothed outcomes (RKS) in Chernozhukov et al. 2022b):* Let $g(t, s) = \mathbb{E}\left[K_r\left(\frac{T-t}{h_r}\right)\Big|S = s\right]$. We obtain $\widehat{p}_{T|S}(t|s) = \widehat{g}(t, s)$ by regressing $\left\{K_r\left(\frac{T_i-t}{h_r}\right)\right\}_{i=1}^{n}$ against $\{S_i\}_{i=1}^{n}$ via any machine learning method. Note that

$$g(t, s) = \mathbb{E}\left[K_r\left(\frac{T-t}{h_r}\right)\Big|S = s\right] \to p_{T|S}(t|s) \quad \text{as} \quad h_r \to 0,$$

where $K_r : \mathbb{R} \to [0, \infty)$ is a kernel function and $h_r > 0$ is a bandwidth parameter.

Localized RA Derivative Estimator for $\theta(t)$ Without Positivity

Combining two nuisance function estimators $\widehat{\beta}(t, \boldsymbol{s})$ and $\widehat{F}_{\boldsymbol{S}|T}(\boldsymbol{s}|t)$, we derive our **localized RA derivative estimator** of $\theta(t)$ with kernel smoothing as:

$$\widehat{\theta}_{\mathrm{C,RA}}(t) = \int \widehat{\beta}(t, \boldsymbol{s})\, d\widehat{P}_{\hbar}(\boldsymbol{s}|t) = \frac{\sum_{i=1}^{n} \widehat{\beta}(t, \boldsymbol{S}_i) \cdot \bar{K}_T\left(\frac{T_i - t}{\hbar}\right)}{\sum_{j=1}^{n} \bar{K}_T\left(\frac{T_j - t}{\hbar}\right)}.$$

Our **RA integral estimator** takes the form

$$\widehat{m}_{\mathrm{C,RA}}(t) = \frac{1}{n} \sum_{i=1}^{n} \left[Y_i + \int_{\tilde{t}=T_i}^{\tilde{t}=t} \widehat{\theta}_{\mathrm{C,RA}}(\tilde{t})\, d\tilde{t} \right].$$

- $\widehat{m}_{\mathrm{C,RA}}(t)$, under our kernel smoothing-based estimators, is a *linear smoother*.
- We can also fit $\mu(t, \boldsymbol{s})$ via neural networks and obtain an estimator for $\beta(t, \boldsymbol{s})$ via automatic differentiation in `PyTorch` (Paszke et al., 2017).

▶ **Issue:** The integral in $\widehat{m}_{\mathrm{C,RA}}(t)$ could be analytically difficult to compute.

Our integral estimator takes the form

$$\widehat{m}_{\mathrm{C,RA}}(t) = \frac{1}{n}\sum_{i=1}^{n}\left[Y_i + \int_{\tilde{t}=T_i}^{\tilde{t}=t} \widehat{\theta}_{\mathrm{C,RA}}(\tilde{t})\, d\tilde{t}\right].$$

▶ **Riemann Sum Approximation:** Let $T_{(1)} \leq \cdots \leq T_{(n)}$ be the order statistics of $T_1, ..., T_n$ and $\Delta_j = T_{(j+1)} - T_{(j)}$ for $j = 1, ..., n-1$.

- Approximate $\widehat{m}_{\mathrm{C,RA}}(T_{(j)})$ for each $j = 1, ..., n$ as:

$$\widehat{m}_{\mathrm{C,RA}}(T_{(j)}) \approx \frac{1}{n}\sum_{i=1}^{n} Y_i + \frac{1}{n}\sum_{i=1}^{n-1}\Delta_i\left[i \cdot \widehat{\theta}_{\mathrm{C,RA}}(T_{(i)})\mathbb{1}_{\{i<j\}} - (n-i)\cdot \widehat{\theta}_{\mathrm{C,RA}}(T_{(i+1)})\mathbb{1}_{\{i\geq j\}}\right].$$

- Evaluate $\widehat{m}_{\mathrm{C,RA}}(t)$ at any $t \in \left[T_{(j)}, T_{(j+1)}\right]$ by a linear interpolation between $\widehat{m}_{\mathrm{C,RA}}(T_{(j)})$ and $\widehat{m}_{\mathrm{C,RA}}(T_{(j+1)})$.
- The approximation error $O_P\left(\frac{1}{n}\right)$ is asymptotically negligible.

1. Compute $\widehat{m}_{C,RA}(t)$ on the original data $\{(Y_i, T_i, S_i)\}_{i=1}^n$.
2. Generate B bootstrap samples $\left\{\left(Y_i^{*(b)}, T_i^{*(b)}, S_i^{*(b)}\right)\right\}_{i=1}^n$ by sampling with replacement and compute $\widehat{m}_{C,RA}^{*(b)}(t)$ for each $b = 1, \ldots, B$.
3. Let $\alpha \in (0,1)$ be a pre-specified significance level.
   - For pointwise inference at $t_0 \in \mathcal{T}$, calculate the $1-\alpha$ quantile $\zeta_{1-\alpha}^*(t_0)$ of $\{D_1(t_0), \ldots, D_B(t_0)\}$, where $D_b(t_0) = \left|\widehat{m}_{C,RA}^{*(b)}(t_0) - \widehat{m}_{C,RA}(t_0)\right|$ for $b = 1, \ldots, B$.
   - For uniform inference on $m(t)$, compute the $1-\alpha$ quantile $\xi_{1-\alpha}^*$ of $\{D_{\sup,1}, \ldots, D_{\sup,B}\}$, where $D_{\sup,b} = \sup_{t\in\mathcal{T}} \left|\widehat{m}_{C,RA}^{*(b)}(t) - \widehat{m}_{C,RA}(t)\right|$ for $b = 1, \ldots, B$.
4. Define the $1-\alpha$ confidence interval for $m(t_0)$ as:

$$\left[\widehat{m}_{C,RA}(t_0) - \zeta_{1-\alpha}^*(t_0),\ \widehat{m}_{C,RA}(t_0) + \zeta_{1-\alpha}^*(t_0)\right]$$

and the simultaneous $1-\alpha$ confidence band for every $t \in \mathcal{T}$ as:

$$\left[\widehat{m}_{C,RA}(t) - \xi_{1-\alpha}^*,\ \widehat{m}_{C,RA}(t) + \xi_{1-\alpha}^*\right].$$

Multiplier Bootstrap With $\widehat{\theta}_{\mathrm{DR}}(t)$ for Uniform Inference

Let $\{Z_i\}_{i=1}^n$ be a sequence of i.i.d. random variables independent of the observed data $\{(Y_i, T_i, \mathbf{S}_i)\}_{i=1}^n$ with $\mathbb{E}(Z_i) = \mathrm{Var}(Z_i) = 1$ and sub-exponential tails (Fan et al., 2022; Colangelo and Lee, 2020).

1. Sample B different i.i.d. datasets $\left\{Z_i^{(b)}\right\}_{i=1}^n, b = 1, ..., B$.
2. Compute the bootstrap DR estimators for $\theta(t)$ for $b = 1, ..., B$ as:

$$\widehat{\theta}_{\mathrm{DR}}^{(b)*}(t) = \frac{1}{nh}\sum_{i=1}^n Z_i^{(b)} \left\{ \frac{\left(\frac{T_i - t}{h}\right) K\left(\frac{T_i - t}{h}\right)}{h \cdot \kappa_2 \cdot \widehat{p}_{T|S}(T_i|\mathbf{S}_i)} \left[Y_i - \widehat{\mu}(t, \mathbf{S}_i) - (T_i - t)\cdot \widehat{\beta}(t, \mathbf{S}_i)\right] + h \cdot \widehat{\beta}(t, \mathbf{S}_i) \right\}.$$

3. Let $\widehat{Q}(1-\tau)$ be the $(1-\tau)$ quantile of the sequence $\left\{\sup_{t\in\mathcal{T}} \sqrt{nh^3} \left|\frac{\widehat{\theta}_{\mathrm{DR}}^{(b)*}(t) - \widehat{\theta}_{\mathrm{DR}}(t)}{\sqrt{\widehat{V}_{\theta}(t)}}\right|\right\}_{b=1}^B$.
4. The uniform $(1-\tau)$-level confidence band of $\theta(t)$ is given by

$$\left[\widehat{\theta}_{\mathrm{DR}}(t) \pm \widehat{Q}(1-\tau)\sqrt{\frac{\widehat{V}_{\theta}(t)}{nh^3}}\right].$$

▶ The self-normalizing technique can reduce the instability of IPW and DR estimators (Kallus and Zhou, 2018):

1. **Self-Normalized Estimators Under Positivity:**

$$\widehat{\theta}_{\mathrm{IPW}}^{\mathrm{norm}}(t) = \frac{\widehat{\theta}_{\mathrm{IPW}}(t)}{\frac{1}{nh}\sum_{j=1}^{n}\frac{K\left(\frac{T_j-t}{h}\right)}{\widehat{p}_{T|S}(T_j|S_j)}} = \frac{\sum_{i=1}^{n}\frac{Y_i\left(\frac{T_i-t}{h}\right)K\left(\frac{T_i-t}{h}\right)}{\widehat{p}_{T|S}(T_i|S_i)}}{\kappa_2\cdot h\sum_{j=1}^{n}\frac{K\left(\frac{T_j-t}{h}\right)}{\widehat{p}_{T|S}(T_j|S_j)}},$$

and

$$\widehat{\theta}_{\mathrm{DR}}^{\mathrm{norm}}(t) = \frac{\sum_{i=1}^{n}\frac{\left[Y_i-\widehat{\mu}(t,S_i)-(T_i-t)\cdot\widehat{\beta}(t,S_i)\right]\left(\frac{T_i-t}{h}\right)K\left(\frac{T_i-t}{h}\right)}{\widehat{p}_{T|S}(T_i|S_i)}}{\kappa_2\cdot h\sum_{j=1}^{n}\frac{K\left(\frac{T_j-t}{h}\right)}{\widehat{p}_{T|S}(T_j|S_j)}} + \frac{1}{n}\sum_{i=1}^{n}\widehat{\beta}(t,S_i).$$

2. **Self-Normalized Estimators Without Positivity:**

$$\widehat{\theta}^{\text{norm}}_{\text{C,IPW}}(t) = \frac{\widehat{\theta}_{\text{C,IPW}}(t)}{\frac{1}{nh}\sum_{j=1}^{n}\frac{K\left(\frac{T_j-t}{h}\right)\cdot\widehat{p}_{\zeta}(S_j|t)}{\widehat{p}(T_j,S_j)}} = \frac{\sum_{i=1}^{n}\frac{Y_i\cdot\left(\frac{T_i-t}{h}\right)K\left(\frac{T_i-t}{h}\right)\cdot\widehat{p}_{\zeta}(S_i|t)}{\widehat{p}(T_i,S_i)}}{\kappa_2\cdot h\sum_{j=1}^{n}\frac{K\left(\frac{T_j-t}{h}\right)\cdot\widehat{p}_{\zeta}(S_j|t)}{\widehat{p}(T_j,S_j)}},$$

and

$$\widehat{\theta}^{\text{norm}}_{\text{C,DR}}(t) = \frac{\sum_{i=1}^{n}\frac{\left[Y_i-\widehat{\mu}(t,S_i)-(T_i-t)\cdot\widehat{\beta}(t,S_i)\right]\left(\frac{T_i-t}{h}\right)K\left(\frac{T_i-t}{h}\right)\cdot\widehat{p}_{\zeta}(S_i|t)}{\widehat{p}(T_i,S_i)}}{\kappa_2\cdot h\sum_{j=1}^{n}\frac{K\left(\frac{T_j-t}{h}\right)\cdot\widehat{p}_{\zeta}(S_j|t)}{\widehat{p}(T_j,S_j)}} + \int \widehat{\beta}(t,s)\cdot\widehat{p}_{\zeta}(s|t)\,ds.$$

Assumption (Differentiability of the conditional mean outcome function)

For any $(t, \boldsymbol{s}) \in \mathcal{T} \times \mathcal{S}$ *and* $\mu(t, \boldsymbol{s}) = \mathbb{E}(Y|T = t, \boldsymbol{S} = \boldsymbol{s})$*, it holds that*

1. $\mu(t, \boldsymbol{s})$ *is at least four times continuously differentiable with respect to* t*.*
2. $\mu(t, \boldsymbol{s})$ *and all of its partial derivatives are uniformly bounded on* $\mathcal{T} \times \mathcal{S}$*.*
3. *There exist absolute constants* $\sigma, A_0 > 0$ *such that* $\mathrm{Var}(Y|T = t, \boldsymbol{S} = \boldsymbol{s}) = \sigma^2$ *and* $\mathbb{E}|Y|^4 < A_0 < \infty$ *uniformly in* $\mathcal{J}$*.*

Let $\mathcal{J}$ be the support of the joint density $p(t, \boldsymbol{s})$.

Assumption (Differentiability of the density functions)

For any $(t, \boldsymbol{s}) \in \mathcal{J}$, it holds that

1. *The joint density $p(t, \boldsymbol{s})$ and the conditional density $p_{T|\boldsymbol{S}}(t|\boldsymbol{s})$ are at least three times continuously differentiable with respect to t.*
2. *$p(t, \boldsymbol{s})$, $p_{T|\boldsymbol{S}}(t|\boldsymbol{s})$, $p_{\boldsymbol{S}|T}(\boldsymbol{s}|t)$, as well as all of the partial derivatives of $p(t, \boldsymbol{s})$ and $p_{T|\boldsymbol{S}}(t|\boldsymbol{s})$ are bounded and continuous up to the boundary $\partial\mathcal{J}$.*
3. *The support $\mathcal{T}$ of the marginal density $p_T(t)$ is compact and $p_T(t)$ is uniformly bounded away from 0 within $\mathcal{T}$.*

Assumption (Boundary conditions)

1. *There exists some constants* $r_1, r_2 \in (0,1)$ *such that for any* $(t, \mathbf{s}) \in \mathcal{J}$ *and all* $\delta \in (0, r_1]$, *there is a point* $(t', \mathbf{s}') \in \mathcal{J}$ *satisfying*

$$\mathcal{B}\left((t', \mathbf{s}'),\, r_2\delta\right) \subset \mathcal{B}\left((t, \mathbf{s}),\, \delta\right) \cap \mathcal{J},$$

where $\mathcal{B}((t, \mathbf{s}),\, r) = \left\{(t_1, \mathbf{s}_1) \in \mathbb{R}^{d+1} : ||(t_1 - t, \mathbf{s}_1 - \mathbf{s})||_2 \leq r\right\}$ *with* $||\cdot||_2$ *being the standard Euclidean norm.*

2. *For any* $(t, \mathbf{s}) \in \partial\mathcal{J}$, *the boundary of* $\mathcal{J}$, *it satisfies that* $\frac{\partial}{\partial t} p(t, \mathbf{s}) = \frac{\partial}{\partial s_j} p(t, \mathbf{s}) = 0$ *and* $\frac{\partial^2}{\partial s_j^2} \mu(t, \mathbf{s}) = 0$ *for all* $j = 1, ..., d$.

3. *For any* $\delta > 0$, *the Lebesgue measure of the set* $\partial\mathcal{J} \oplus \delta$ *satisfies* $|\partial\mathcal{J} \oplus \delta| \leq A_1 \cdot \delta$ *for some absolute constant* $A_1 > 0$, *where* $\partial\mathcal{J} \oplus \delta = \left\{\mathbf{z} \in \mathbb{R}^{d+1} : \inf_{\mathbf{x} \in \partial\mathcal{J}} ||\mathbf{z} - \mathbf{x}||_2 \leq \delta\right\}$.

Detailed Regularity Assumptions

Assumption (Regular kernel conditions)

A kernel function $K : \mathbb{R} \to [0, \infty)$ *is bounded and compactly supported on* $[-1, 1]$ *with* $\int_{\mathbb{R}} K(t)\, dt = 1$ *and* $K(t) = K(-t)$*. In addition, it holds that*

1. $\kappa_j := \int_{\mathbb{R}} u^j K(u)\, du < \infty$ *and* $\nu_j := \int_{\mathbb{R}} u^j K^2(u)\, du < \infty$ *for all* $j = 1, 2, \ldots$.
2. K *is a second-order kernel, i.e.,* $\kappa_1 = 0$ *and* $\kappa_2 > 0$.
3. $\mathcal{K} = \left\{ t' \mapsto \left(\frac{t'-t}{h}\right)^{k_1} K\left(\frac{t'-t}{h}\right) : t \in \mathcal{T}, h > 0, k_1 = 0, 1 \right\}$ *is a bounded VC-type class of measurable functions on* $\mathbb{R}$.

Assumption (Smoothness condition on $\mathcal{S}(t)$)

For any $\delta \in \mathbb{R}$ *and* $t \in \mathcal{T}$*, there exists an absolute constant* $A_0 > 0$ *such that either of the following holds true:*

(i) " $\mathcal{S}(t) \ominus (A_0|\delta|) \subset \mathcal{S}(t+\delta)$ *" for the support shrinking approach;*

(ii) " $\mathcal{L}_{A_0|\delta|}(t) \subset \mathcal{S}(t+\delta)$ *" for the level set approach.*

- The support $\mathcal{J}$ of $(T, \mathbf{S})$ may not cover $\mathcal{T} \times \mathcal{S}$ without positivity.
- The localized derivative estimator $\widehat{\theta}_C(t) = \int \widehat{\beta}(t, \mathbf{s})\, d\widehat{\mathrm{P}}_{\hbar}(\mathbf{s}|t)$ only requires $\widehat{\beta}(t, \mathbf{s})$ to be consistent in $\mathcal{J}$.

Lemma (Lemma 3 in Zhang et al. 2024)

Under some regularity conditions, as $h, b, \frac{\max\{h,b\}^4}{h} \to 0$ *and* $\frac{|\log(hb^d)|}{nh^3b^d} \to \infty$,

$$\sup_{(t,\mathbf{s})\in\mathcal{J}} \left|\widehat{\beta}(t, \mathbf{s}) - \frac{\partial}{\partial t}\mu(t, \mathbf{s})\right| = O\left(h^q + b^2 + \frac{\max\{h, b\}^4}{h}\right) + O_P\left(\sqrt{\frac{|\log(hb^d)|}{nh^3b^d}}\right).$$

Combining with the consistency of $\widehat{\mathrm{P}}_{\hbar}(\boldsymbol{s}|t)$ via the technique in Fan et al. (1998), we have the following results.

Theorem (Theorem 4 in Zhang et al. 2024)

Under some regularity conditions, when $q = 2$ *and* $h, b, \hbar, \frac{\max\{h,b\}^4}{h} \to 0$ *and* $\frac{n \max\{h,\hbar\} b^d}{\log n}, \frac{n\hbar}{\log n} \to \infty$,

$$\sup_{t \in \mathcal{T}} \left|\widehat{\theta}_{\mathrm{C,RA}}(t) - \theta_{\mathrm{C}}(t)\right| = \underbrace{O\left(h^2 + b^2 + \frac{\max\{b,h\}^4}{h}\right)}_{\text{Bias term}} + \underbrace{O_P\left(\sqrt{\frac{\log n}{nh^3}} + \hbar^2 + \sqrt{\frac{\log n}{n\hbar}}\right)}_{\text{Stochastic variation}},$$

$$\sup_{t \in \mathcal{T}} \left|\widehat{m}_{\mathrm{C,RA}}(t) - m(t)\right| = O\left(h^2 + b^2 + \frac{\max\{b,h\}^4}{h}\right) + O_P\left(\frac{1}{\sqrt{n}} + \sqrt{\frac{\log n}{nh^3}} + \hbar^2 + \sqrt{\frac{\log n}{n\hbar}}\right).$$

$$\sup_{t\in\mathcal{T}} \left|\widehat{m}_{\mathrm{C,RA}}(t) - m(t)\right| = O\left(h^2 + b^2 + \frac{\max\{b,h\}^4}{h}\right) + O_P\left(\frac{1}{\sqrt{n}} + \sqrt{\frac{\log n}{nh^3}} + \hbar^2 + \sqrt{\frac{\log n}{n\hbar}}\right).$$

- Blue term: the estimation bias of local polynomial estimator $\widehat{\beta}(t,\boldsymbol{s}) = \widehat{\beta}_2(t,\boldsymbol{s})$.
- Orange term: additional bias of $\widehat{\beta}_2(t,\boldsymbol{s})$ at the boundary $\partial\mathcal{J}$.
- Teal term: asymptotic rate from $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^n Y_i$.
- Red term: stochastic variation of $\widehat{\beta}_2(t,\boldsymbol{s})$.
- Cyan term: asymptotic rate from the Nadaraya-Watson conditional CDF estimator $\widehat{\mathrm{P}}_{\hbar}(\boldsymbol{s}|t)$.

Lemma (Lemma 5 in Zhang et al. 2024)

Under the same regularity conditions, if $h \asymp n^{-\frac{1}{\gamma}}$ *and* $\hbar \asymp n^{-\frac{1}{\varpi}}$ *for some* $\gamma \geq \varpi > 0$ *such that* $\frac{nh^5}{\log n} \to c_1$ *and* $\frac{n\hbar^5}{\log n} \to c_2$ *for some* $c_1, c_2 \geq 0$ *and* $\frac{n \max\{h,\hbar\}b^d}{\log n}, \frac{n\hbar}{\log n}, \frac{h^3 \log n}{\hbar}, \frac{nh^3\hbar^4}{\log n} \to \infty$ *as* $n \to \infty$, *then for any* $t \in \mathcal{T}$,

$$\sqrt{nh^3}\left[\widehat{\theta}_{\mathrm{C,RA}}(t) - \theta(t)\right] = \mathbb{G}_n \bar{\varphi}_t + o_P(1) \quad \text{and} \quad \sqrt{nh^3}\left[\widehat{m}_{\mathrm{C,RA}}(t) - m(t)\right] = \mathbb{G}_n \varphi_t + o_P(1),$$

where

$$\bar{\varphi}_t(Y, T, \mathbf{S}) = \frac{C_{K_T}\left[Y - \mu(T, \mathbf{S})\right]}{\sqrt{h} \cdot p_T(t)} \left(\frac{T-t}{h}\right) K_T\left(\frac{T-t}{h}\right)$$

and $\varphi_t(Y, T, \mathbf{S}) = \mathbb{E}_{T_1}\left[\int_{T_1}^{t} \bar{\varphi}_{\tilde{t}}(Y, T, \mathbf{S})\, d\tilde{t}\right]$ *with* $\mathbb{G}_n = \sqrt{n}\left(\mathbb{P}_n - \mathrm{P}\right)$, *where* $C_{K_T} > 0$ *is a constant that only depends on* K_T.

▶ **Note:** $\bar{\varphi}_t$ and φ_t are the IPW components of the *approximated* efficient influence functions.

Theorem (Theorems 6 and 7 in Zhang et al. 2024)

Under the same regularity conditions, if $h \asymp n^{-\frac{1}{\gamma}}$ *and* $b \lesssim \hbar \asymp n^{-\frac{1}{\varpi}}$ *for some* $\gamma \geq \varpi > 0$ *such that* $\frac{nh^{d+5}}{\log n} \to c_1$ *and* $\frac{n\hbar^5}{\log n} \to c_2$ *for some* $c_1, c_2 \geq 0$ *and* $\frac{\hbar}{h^3 \log n}, \hbar n^{\frac{1}{3}} \log n, \frac{\sqrt{n\hbar}}{\log n}, \frac{n \max\{h,\hbar\} b^d}{\log n} \to \infty$ *as* $n \to \infty$*,*

1. $$\left|\sqrt{nh^3} \sup_{t\in\mathcal{T}} \left|\widehat{m}_{C,RA}(t) - m(t)\right| - \sup_{t\in\mathcal{T}} \left|\mathbb{G}_n \varphi_t\right|\right| = O_P\left(\sqrt{nh^3 \max\{h,\hbar\}^4} + \sqrt{\frac{h^3 \log n}{\hbar}} + \frac{\log n}{\sqrt{n\hbar}} + \sqrt{\frac{\log n}{nb^d \hbar}}\right).$$

2. *there exists a mean-zero Gaussian process* $\mathbb{B}$ *such that*

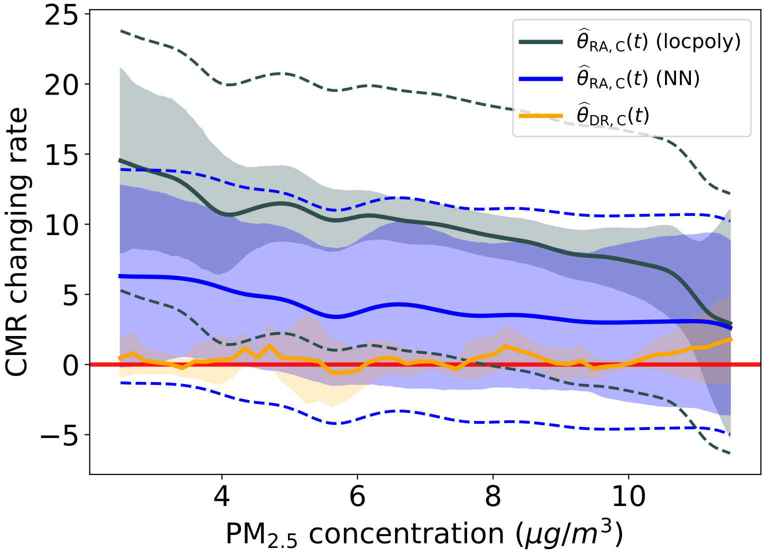
$$\sup_{u\geq 0}\left|P\left(\sqrt{nh^3} \sup_{t\in\mathcal{T}} \left|\widehat{m}_{C,RA}(t) - m(t)\right| \leq u\right) - P\left(\sup_{f\in\mathcal{F}} |\mathbb{B}(f)| \leq u\right)\right| = O\left(\left(\frac{\log^5 n}{nh^3}\right)^{\frac{1}{8}} + \left(\frac{\log^2 n}{nb^d \hbar}\right)^{\frac{3}{8}}\right).$$

3. $$\sup_{u\geq 0}\left|P\left(\sqrt{nh^3} \sup_{t\in\mathcal{T}} \left|\widehat{m}^*_{C,RA}(t) - \widehat{m}_{C,RA}(t)\right| \leq u \,\middle|\, \mathbb{U}_n\right) - P\left(\sup_{f\in\mathcal{F}} |\mathbb{B}(f)| \leq u\right)\right| = O_P\left(\left(\frac{\log^5 n}{nh^3}\right)^{\frac{1}{8}} + \left(\frac{\log^2 n}{nb^d \hbar}\right)^{\frac{3}{8}}\right),$$

where $\mathcal{F} = \{(v, x, z) \mapsto \varphi_t(v, x, z) : t \in \mathcal{T}\}$.

1. $\mathcal{F}$ is not Donsker because φ_t is not uniformly bounded as $h \to 0$.
   - However, $\tilde{\mathcal{F}} = \left\{(v, x, z) \mapsto \sqrt{h^3} \cdot \varphi_t(v, x, z) : t \in \mathcal{T}\right\}$ is of VC-type.
   - Gaussian approximation in Chernozhukov et al. (2014) can be applied to bound the difference between $\sup_{f \in \mathcal{F}} |\mathbb{G}_n(f)|$ and $\sup_{f \in \mathcal{F}} |\mathbb{B}(f)|$.
2. As long as $\mathrm{Var}(Y|T = t, \boldsymbol{S} = \boldsymbol{s}) \geq \sigma^2 > 0$, $\mathrm{Var}\left[\varphi_t(Y, T, \boldsymbol{S})\right]$ is a positive finite number.
   - The asymptotic linearity (or V-statistic) is non-degenerate.
   - Pointwise bootstrap confidence intervals are asymptotically valid.
3. For the validity of uniform bootstrap confidence band, one can choose the bandwidths $h \asymp \hbar = O\left(n^{-\frac{1}{5}}\right)$ and $\left(\frac{\log n}{n}\right)^{\frac{4}{5d}} \lesssim b \lesssim n^{-\frac{1}{5}}$.
   - These orders align with the outputs from the usual bandwidth selection methods (Bashtannyk and Hyndman, 2001; Li and Racine, 2004).
   - No explicit undersmoothing is required!!

Additional Results for $PM_{2.5}$ on CMRs



Shaded areas: 95% pointwise confidence intervals; **Regions between dashed lines:** 95% uniform confidence bands.

Simulation Setup for Estimating $m(t)$ and $\theta(t)$ Without Positivity

- Use the Epanechnikov kernel for K_T and K_S (with the product kernel technique) and Gaussian kernel for $\bar{K}_T$.
- Select the bandwidth parameters $h, b > 0$ by modifying the rule-of-thumb method in Yang and Tschernig (1999).
- Set the bandwidth parameter $\hbar > 0$ to the normal reference rule in Chacón et al. (2011); Chen et al. (2016).
- Set the bootstrap resampling time $B = 1000$ and the nominal level for confidence intervals or bands to 95%.
- Compare our proposed estimators with the regression adjustment estimators under the same choices of bandwidth parameters:

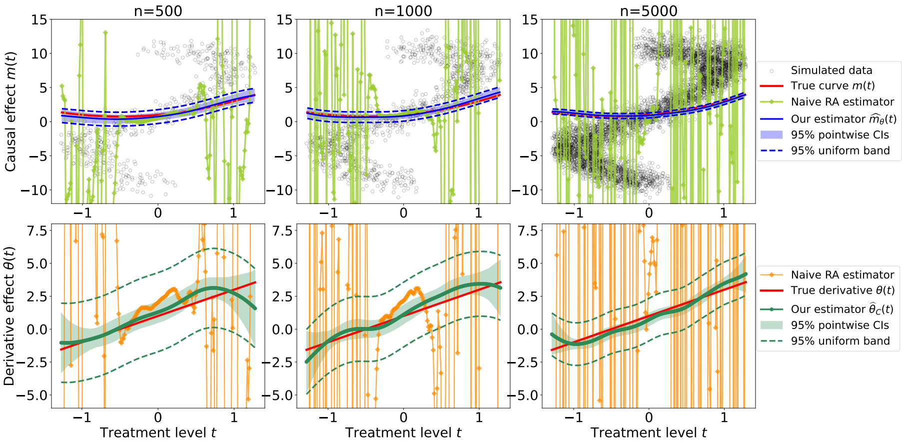
$$\widehat{m}_{\mathrm{RA}}(t) = \frac{1}{n}\sum_{i=1}^{n} \widehat{\mu}(t, S_i) \quad \text{and} \quad \widehat{\theta}_{\mathrm{RA}}(t) = \frac{1}{n}\sum_{i=1}^{n} \widehat{\beta}(t, S_i).$$

Single Confounder Model Without Positivity

Generate i.i.d. observations $\{(Y_i, T_i, S_i)\}_{i=1}^{2000}$ from

$$Y = T^2 + T + 1 + 10S + \epsilon, \quad T = \sin(\pi S) + E, \quad \text{and} \quad S \sim \text{Uniform}[-1, 1].$$

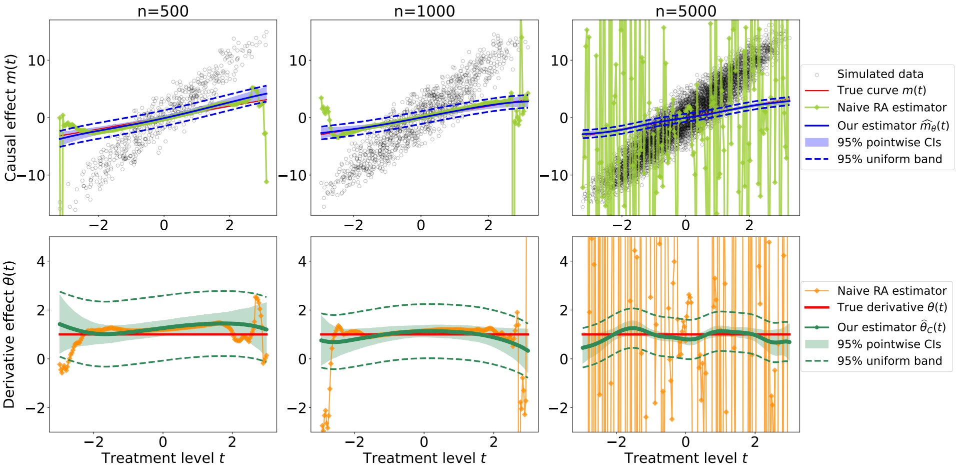
- $E \sim \text{Uniform}[-0.3, 0.3]$ is an independent treatment variation,
- $\epsilon \sim \mathcal{N}(0, 1)$ is an exogenous normal noise.



Generate i.i.d. observations $\{(Y_i, T_i, S_i)\}_{i=1}^{2000}$ from

$$Y = T + 6S_1 + 6S_2 + \epsilon, \quad T = 2S_1 + S_2 + E, \quad \text{and} \quad (S_1, S_2) \sim \text{Uniform}[-1,1]^2,$$

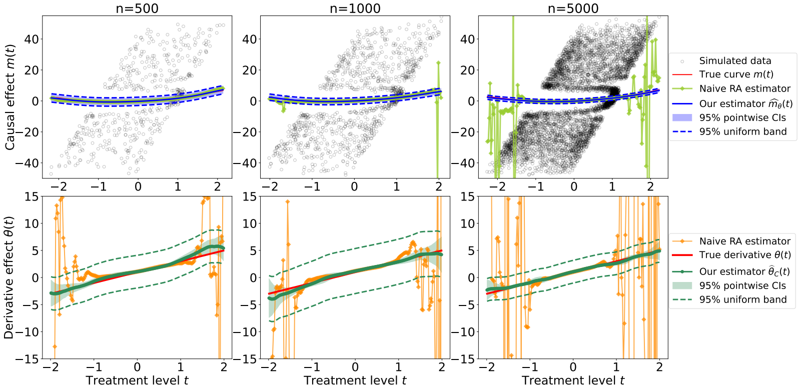
- $E \sim \text{Uniform}[-0.5, 0.5]$ and $\epsilon \sim \mathcal{N}(0,1)$.



Generate i.i.d. observations $\{(Y_i, T_i, S_i)\}_{i=1}^{2000}$ from

$$Y = T^2 + T + 10Z + \epsilon, \quad T = \cos\left(\pi Z^3\right) + \frac{Z}{4} + E, \quad \text{and} \quad Z = 4S_1 + S_2,$$

- $(S_1, S_2) \sim \text{Uniform}[-1, 1]^2$, $E \sim \text{Uniform}[-0.1, 0.1]$, and $\epsilon \sim \mathcal{N}(0, 1)$.
- Those doubly robust methods based on pseudo-outcomes (Kennedy et al., 2017; Takatsu and Westling, 2024) do not work in this example.



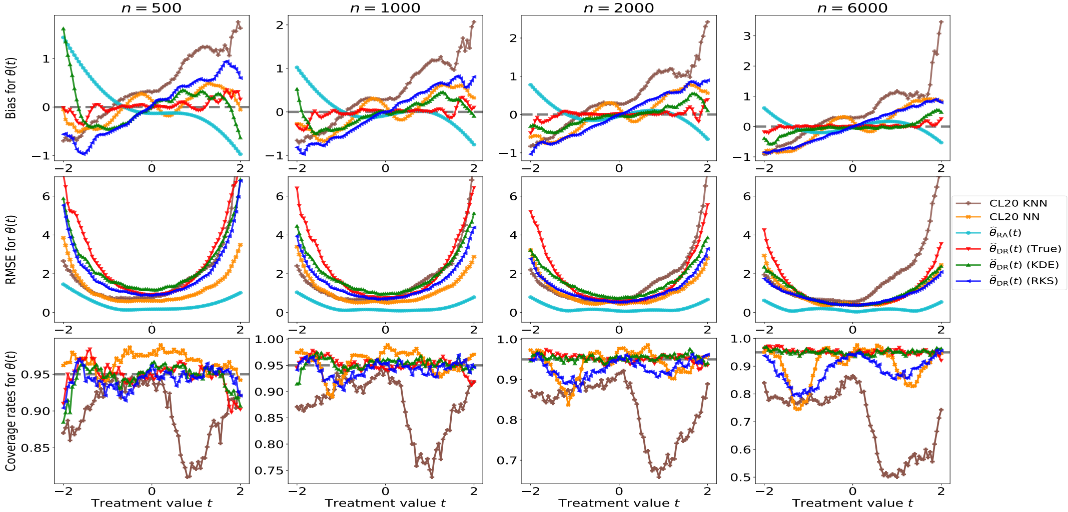
We generate i.i.d. observations $\{(Y_i, T_i, \boldsymbol{S}_i)\}_{i=1}^n$ from the following data-generating model (Colangelo and Lee, 2020):

$$Y = 1.2\,T + T^2 + TS_1 + 1.2\,\boldsymbol{\xi}^T\boldsymbol{S} + \epsilon\sqrt{0.5 + F_{\mathcal{N}(0,1)}(S_1)}, \quad \epsilon \sim \mathcal{N}(0,1),$$
$$T = F_{\mathcal{N}(0,1)}\left(3\boldsymbol{\xi}^T\boldsymbol{S}\right) - 0.5 + 0.75E, \quad \boldsymbol{S} = (S_1, ..., S_d)^T \sim \mathcal{N}_d\left(\boldsymbol{0}, \Sigma\right),\ E \sim \mathcal{N}(0,1),$$

where

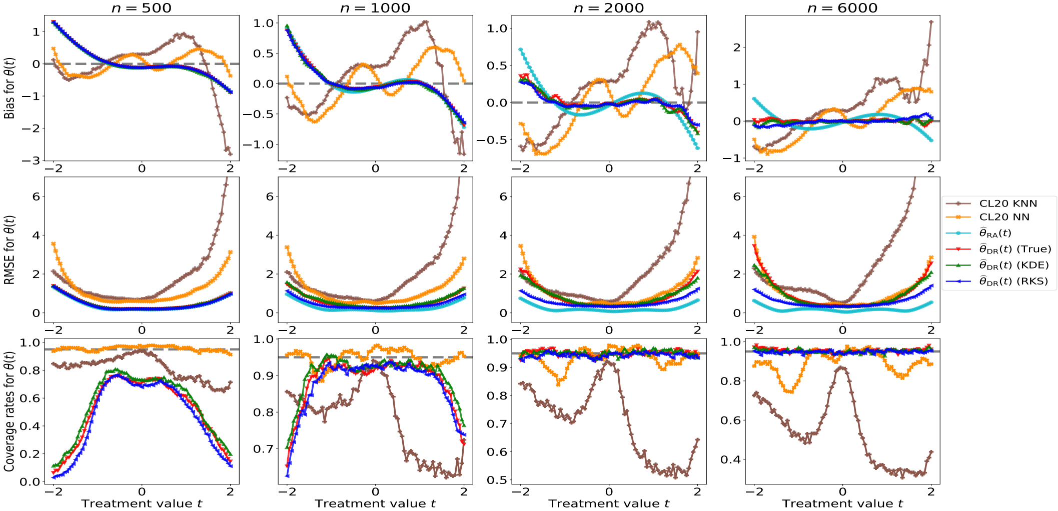
- $F_{\mathcal{N}(0,1)}$ is the CDF of $\mathcal{N}\left(0,1\right)$ and $d = 20$.
- $\boldsymbol{\xi} = (\xi_1, ..., \xi_d)^T \in \mathbb{R}^d$ has its entry $\xi_j = \frac{1}{j^2}$ for $j = 1, ..., d$ and $\Sigma_{ii} = 1$, $\Sigma_{ij} = 0.5$ when $|i-j| = 1$, and $\Sigma_{ij} = 0$ when $|i-j| > 1$ for $i, j = 1, ..., d$.
- The dose-response curve is given by $m(t) = 1.2t + t^2$, and our parameter of interest is the derivative effect curve $\theta(t) = 1.2 + 2t$.

Simulations for Estimating $\theta(t)$ Under Positivity



Comparisons between our proposed estimators and the finite-difference approaches by Colangelo and Lee (2020) ("CL20") under positivity and **with 5-fold cross-fitting** across various sample sizes.

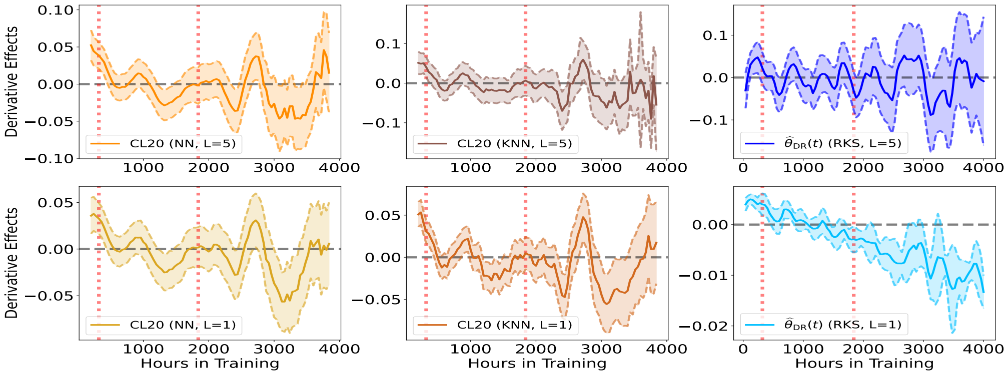
Simulations for Estimating $\theta(t)$ Under Positivity



Comparisons between our proposed estimators and the finite-difference approaches by Colangelo and Lee (2020) ("CL20") under positivity and **without cross-fitting** across various sample sizes.

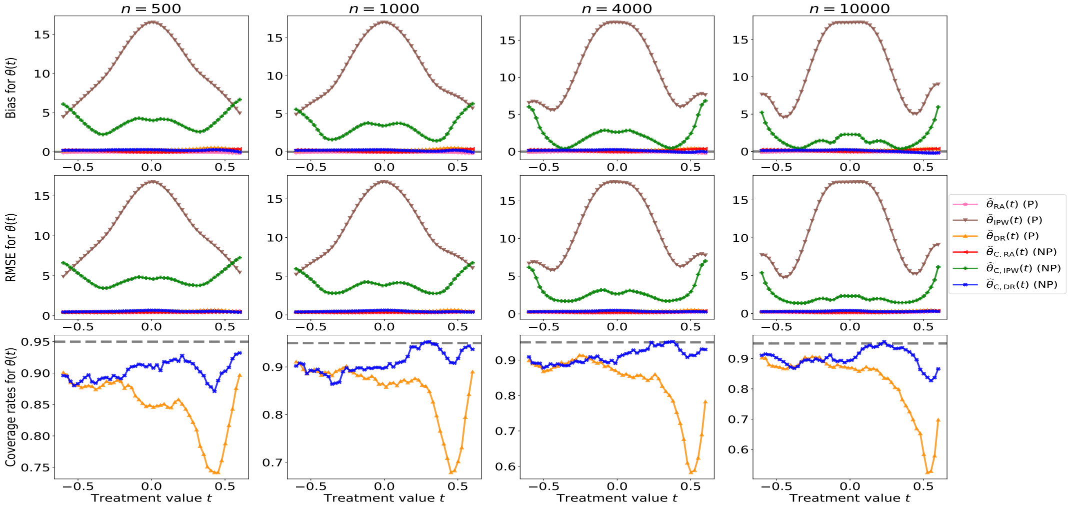
We compare our proposed DR estimator $\widehat{\theta}_{\mathrm{DR}}(t)$ under positivity with the finite-difference method (Colangelo and Lee 2020; CL20) on the U.S. Job Corps program (Schochet et al., 2001).

- Y is the proportion of weeks employed in 2^{nd} year after enrollment.
- T is the total hours of academic and vocational training received.
- $\boldsymbol{S}$ comprises 49 socioeconomic characteristics, and $n = 4024$.



Simulations for $\widehat{\theta}_{C,RA}(t)$, $\widehat{\theta}_{C,IPW}(t)$, $\widehat{\theta}_{C,DR}(t)$ Without Positivity

$$Y = T^3 + T^2 + 10S + \epsilon, \quad T = \sin(\pi S) + E, \quad S \sim \text{Unif}[-1, 1],\ E \sim \text{Unif}[-0.3, 0.3].$$



- **Note:** $\beta(t,s) = \frac{\partial}{\partial t}\mu(t,s)$ is estimated via automatic differentiation of a well-trained neural network (inspired by Luedtke 2024).